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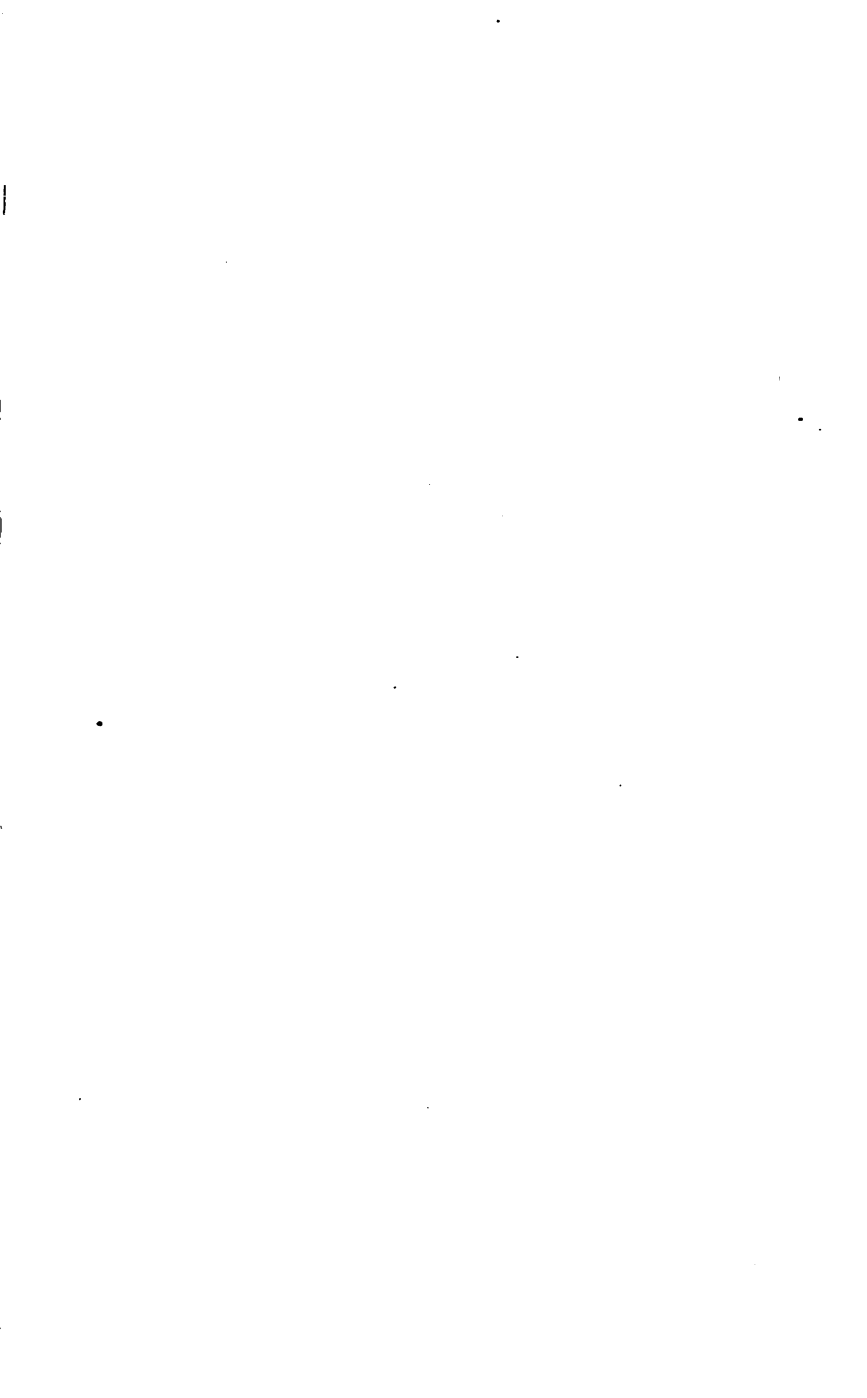
## Connections

Pages 12, 81, 137, ~~139~~, 151  
169, 98.

## Answers

Pages 6, 23, 25,  
21





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**Special Terms and Circular on Application.**

ELEMENTS  
OF  
ANALYTIC GEOMETRY.

BY  
G. A. WENTWORTH, A.M.,  
PROFESSOR OF MATHEMATICS IN PHILLIPS EXETER ACADEMY.



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## PREFACE.

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THIS book is intended for beginners. As beginners generally find great difficulty in comprehending the connection between a locus and its equation, the opening chapter is devoted mainly to an attempt, by means of easy illustrations and examples, to make this connection clear.

Each chapter abounds in exercises; for it is only by solving problems which require some degree of original thought that any real mastery of the study can be gained.

The more difficult propositions have been put at the ends of the chapters, under the heading of "Supplementary Propositions." This arrangement makes it possible for every teacher to mark out his own course. The simplest course will be Chapters I.-III. and Chapters V.-VII., with Review Exercises and Supplementary Propositions left out. Between this course and the entire work the teacher can exercise his choice, and take just so much as time and circumstances will allow.

The author has gathered his materials from many sources, but he is particularly indebted to the English treatise of CHARLES SMITH. Special acknowledgment is due to G. A. HILL, A.M., of Cambridge, Mass., and to Prof. J. M. TAYLOR, Madison University, Hamilton, New York, for assistance in the preparation of the book.

Corrections and suggestions will be thankfully received.

G. A. WENTWORTH.

PHILLIPS EXETER ACADEMY,  
January, 1888.



## NOTE TO SECOND EDITION.

IN this edition such changes have been made as actual experience in the class-room has shown to be desirable.

A chapter on Higher Plane Curves, and four chapters on Solid Geometry have been added, making the work sufficiently extensive for our best schools and colleges.

An effort has been made to have this edition free from errors. It is not likely, however, that this effort has been entirely successful, and the author will be very grateful to any reader who will notify him of any needed corrections.

G. A. WENTWORTH.



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# ANALYTIC GEOMETRY.

## PART I. — PLANE GEOMETRY.

### CHAPTER I.

#### LOCI AND THEIR EQUATIONS.

##### RECTILINEAR SYSTEM OF CO-ORDINATES.

1. Let  $XX'$  and  $YY'$  (Fig. 1) be two fixed lines intersecting in the point  $O$ . These lines divide the plane in which they lie into four portions.

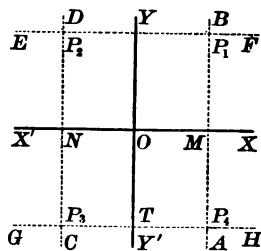


Fig. 1.

Let these parts be called **Quadrants** (as in Trigonometry), and distinguished by naming the area between  $OX$  and  $OY$  the *first* quadrant; that between  $OY$  and  $OX'$  the *second* quadrant; that between  $OX'$  and  $OY'$  the *third* quadrant; and that between  $OY'$  and  $OX$  the *fourth* quadrant.

Suppose the position of a point is described by saying that its distance from  $YY'$ , expressed in terms of some chosen unit of length, is 3, and its distance from  $XX'$  is 4, it being understood that the distance from either line is measured parallel to the other. It is clear that in each quadrant there is one point, and only one, which will satisfy these conditions. The position of the point in each quadrant may be found by drawing parallels to  $YY'$  at the distance 3 from  $YY'$ , and parallels to  $XX'$  at the distance 4 from  $XX'$ ; then the intersections  $P_1$ ,  $P_2$ ,  $P_3$ , and  $P_4$  satisfy the given conditions.

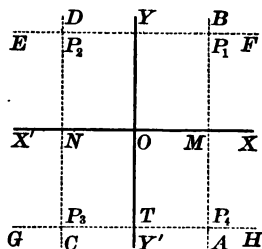


Fig. 1.

2. In order to determine which one of the four points,  $P_1$ ,  $P_2$ ,  $P_3$ ,  $P_4$ , is meant, we adopt the rule that *opposite directions shall be indicated by unlike signs*. As in Trigonometry, distances measured from  $YY'$  to the right are considered *positive*; to the left, *negative*. Distances measured from  $XX'$  upward are *positive*; downward, *negative*. Then the position of  $P_1$  will be denoted by  $+3, +4$ ; of  $P_2$ , by  $-3, +4$ ; of  $P_3$ , by  $-3, -4$ ; of  $P_4$  by  $+3, -4$ .

3. The fixed lines  $XX'$  and  $YY'$  are called the **Axes of Co-ordinates**;  $XX'$  is called the **Axis of Abscissas**, or **Axis of  $x$** ;  $YY'$ , the **Axis of Ordinates**, or **Axis of  $y$** . The intersection  $O$  is called the **Origin**.

The two distances (with signs prefixed) which determine the position of a point are called the **Co-ordinates** of the point; the distance of the point from  $YY'$  is called its **Abscissa**; and the distance from  $XX'$ , its **Ordinate**.

Abscissas are usually denoted by  $x$ , and ordinates by  $y$ , and a point is represented algebraically by simply writing the values of its co-ordinates within parentheses, that of the abscissa being always written first.

Thus  $P_1$  (Fig. 1) is the point  $(3, 4)$ ,  $P_2$  the point  $(-3, 4)$ ,  $P_3$  the point  $(-3, -4)$ , and  $P_4$  the point  $(3, -4)$ . In general the point whose co-ordinates are  $x$  and  $y$  is the point  $(x, y)$ .

4. This method of determining the position of a point in a plane is called the **Rectilinear System of Co-ordinates**. The co-ordinates are called *rectangular*, or *oblique*, according as the axes are rectangular or oblique; that is, according as the axes intersect at right or oblique angles. In the first three chapters we shall use only rectangular co-ordinates.

NOTE. The first man who employed this method successfully in investigating the properties of certain figures was the French philosopher Descartes, whose work on Geometry appeared in the year 1637.

### Ex. 1.

1. What are the co-ordinates of the origin?
2. In what quadrants are the following points ( $a$  and  $b$  being given lengths):  
 $(-a, -b)$ ,  $(-a, b)$ ,  $(a, b)$ ,  $(a, -b)$ .
3. To what quadrants is a point limited if its abscissa is positive? negative? ordinate positive? ordinate negative?
4. In what line does a point lie if its abscissa  $= 0$ ? if its ordinate  $= 0$ ?
5. A point  $(x, y)$  moves parallel to the axis of  $x$ ; which one of its co-ordinates remains constant in value?



6. Construct or *plot* the points:  $(2, 3)$ ,  $(3, -3)$ ,  $(-1, -3)$ ,  $(-4, 4)$ ,  $(3, 0)$ ,  $(-3, 0)$ ,  $(0, 4)$ ,  $(0, -1)$ ,  $(0, 0)$ .

NOTE. To *plot* a point is to mark its proper position on paper, when its co-ordinates are given. The first thing to do is to draw the two axes. The rest of the work is obvious after a study of Nos. 1-3.

7. Construct the triangle whose vertices are the points  $(2, 4)$ ,  $(-2, 7)$ ,  $(-6, -8)$ .

8. Construct the quadrilateral whose vertices are the points  $(7, 2)$ ,  $(0, -9)$ ,  $(-3, -1)$ ,  $(-6, 4)$ .

9. Construct the quadrilateral whose vertices are  $(-3, 6)$ ,  $(-3, 0)$ ,  $(3, 0)$ ,  $(3, 6)$ . What kind of a quadrilateral is it?

10. Mark the four points  $(2, 1)$ ,  $(4, 3)$ ,  $(2, 5)$ , and  $(0, 3)$ , and connect them by straight lines. What kind of a figure do these four lines enclose?

11. The side of a square  $= a$ ; the origin of co-ordinates is the intersection of the diagonals. What are the co-ordinates of the vertices (i.) if the axes are parallel to the sides of the square? (ii.) if the axes coincide with the diagonals?

$$\begin{aligned} \text{Ans. (i.) } & \left(\frac{a}{2}, \frac{a}{2}\right), \left(-\frac{a}{2}, \frac{a}{2}\right), \left(-\frac{a}{2}, -\frac{a}{2}\right), \left(\frac{a}{2}, -\frac{a}{2}\right); \\ \text{(ii.) } & \left(\frac{a}{2}\sqrt{2}, 0\right), \left(0, \frac{a}{2}\sqrt{2}\right), \left(-\frac{a}{2}\sqrt{2}, 0\right), \left(0, -\frac{a}{2}\sqrt{2}\right). \end{aligned}$$

12. The side of an equilateral triangle  $= a$ ; the origin is taken at one vertex and the axis of  $x$  coincides with one side. What are the co-ordinates of the three vertices?

$$\text{Ans. } (0, 0), (a, 0), \left(\frac{a}{2}, \frac{a}{2}\sqrt{3}\right).$$

13. The line joining two points is bisected at the origin. If the co-ordinates of one of the points are  $a$  and  $b$ , what are the co-ordinates of the other?

14. Connect the points  $(5, 3)$  and  $(5, -3)$  by a straight line. What is the direction of this line?

### CIRCULAR MEASURE.

5. In Analytic Geometry, angles are often expressed in degrees, minutes, and seconds; but sometimes it is very convenient to employ the *Circular Measure* of an angle.

In circular measure an angle is defined by the equation

$$\text{angle} = \frac{\text{arc}}{\text{radius}},$$

in which the word "arc" denotes the length of the arc corresponding to the angle when both arc and radius are expressed in terms of a common linear unit.

This equation gives us a correct measure of angular magnitude, because (as shown in Geometry) for a given angle the value of the above ratio of arc and radius is constant for all values of the radius.

If the radius = 1, the equation becomes

$$\text{angle} = \text{arc}; \text{ that is,}$$

*In circular measure an angle is measured by the length of the arc subtended by it in a unit circle.*

It is shown in Geometry that the circumference of a unit circle =  $2\pi$ ; as this circumference contains  $360^\circ$  common measure, the two measures are easily compared by means of the relation

$$360 \text{ degrees} = 2\pi \text{ units, circular measure.}$$

#### Ex. 2.

1. Find the value in circular measure of the angles  $1^\circ$ ,  $45^\circ$ ,  $90^\circ$ ,  $180^\circ$ ,  $270^\circ$ .

$$\text{Ans. } \frac{\pi}{180}, \frac{1}{4}\pi, \frac{1}{2}\pi, \pi, \frac{3}{2}\pi.$$

2. In circular measure, unit angle is that angle whose arc is equal to the radius of the circle. What is the value of this angle in degrees, etc.?

$$\text{Ans. } 57^\circ 17' 45''.$$

## DISTANCE BETWEEN TWO POINTS.

6. To find the distance between two given points:

Let  $P$  and  $Q$  (Fig. 2) be the given points,  $x_1$  and  $y_1$  the co-ordinates of  $P$ ,  $x_2$  and  $y_2$  those of  $Q$ . Also let  $d = PQ =$  the required distance.

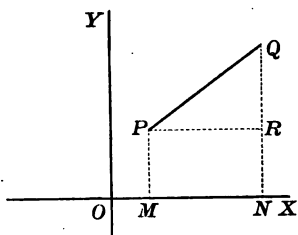


Fig. 2.

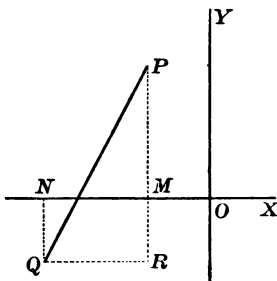


Fig. 3.

Draw  $PM$  and  $QN \parallel$  to  $OY$ , and  $PR \parallel$  to  $OX$ .

Then  $OM = x_1, \quad MP = y_1,$   
 $ON = x_2, \quad NQ = y_2,$   
 $PR = x_2 - x_1, \quad QR = y_2 - y_1.$

By Geometry,

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2;$$

whence

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \quad [1]$$

Since  $(x_1 - x_2)^2 = (x_2 - x_1)^2$ , it makes no difference which point is called  $(x_1, y_1)$ , and which  $(x_2, y_2)$ .

7. Equation [1] is perfectly general, holding true for points situated in any quadrant. Thus, if  $P$  be in the second quadrant and  $Q$  in the third quadrant (Fig. 3),  $x_2 - x_1$  is obviously equal to the leg  $RQ$ ; and since  $y_2$  is negative,  $y_2 - y_1$  is the sum of two negative numbers, and is equal to the absolute length of the leg  $RP$  with the  $-$  sign prefixed.

**NOTE.** The learner should satisfy himself that equation [1] is perfectly general, by constructing other special cases in which the points  $P$  and  $Q$  are in different quadrants. In every case he will find that the numerical values of the expressions  $(x_2 - x_1)$  and  $(y_2 - y_1)$  are the legs of the right triangle the hypotenuse of which is the required distance  $PQ$ .

Equation [1] is merely one illustration of a general truth, of which the learner will gradually become convinced as he proceeds with the study of the subject; namely, that *theorems and formulas deduced by reasoning with points or lines in the first quadrant (where the co-ordinates are always positive) must, from the very nature of the analytic method, hold true when the points or lines are situated in the other quadrants.*

### Ex. 3.

Find the distance

1. From the point  $(-2, 5)$  to the point  $(-8, -3)$ .
2. From the point  $(1, 3)$  to the point  $(6, 15)$ .
3. From the point  $(-4, 5)$  to the point  $(0, 2)$ .
4. From the origin to the point  $(-6, -8)$ .
5. From the point  $(a, b)$  to the point  $(-a, -b)$ .

Find the lengths of the sides of a triangle

6. If the vertices are the points  $(15, -4)$ ,  $(-9, 3)$ ,  $(11, 24)$ .
7. If the vertices are the points  $(2, 3)$ ,  $(4, -5)$ ,  $(-3, -6)$ .
8. If the vertices are the points  $(0, 0)$ ,  $(3, 4)$ ,  $(-3, 4)$ .
9. If the vertices are the points  $(0, 0)$ ,  $(-a, 0)$ ,  $(0, -b)$ .
10. The vertices of a quadrilateral are  $(5, 2)$ ,  $(3, 7)$ ,  $(-1, 4)$ ,  $(-3, -2)$ . Find the lengths of the sides and also of the diagonals.
11. One end of a line whose length is 13 is the point  $(-4, 8)$ ; the ordinate of the other end is 3. What is its abscissa?

12. What equation must the co-ordinates of the point  $(x, y)$  satisfy if its distance from the point  $(7, -2)$  is equal to 11?

13. What equation expresses algebraically the fact that the point  $(x, y)$  is equidistant from the points  $(2, 3)$  and  $(4, 5)$ ?

14. If the value of a quantity depends on the *square* of a length, it is immaterial whether the length be considered positive or negative. Why?

### DIVISION OF A LINE.

8. To bisect the line joining two given points.

Let  $P$  and  $Q$  (Fig. 4) be the given points  $(x_1, y_1)$  and  $(x_2, y_2)$ . Let  $x$  and  $y$  be the co-ordinates of  $R$ , the mid-point of  $PQ$ .

The meaning of the problem is to find the values of  $x$  and  $y$  in terms of  $x_1, y_1$ , and  $x_2, y_2$ .

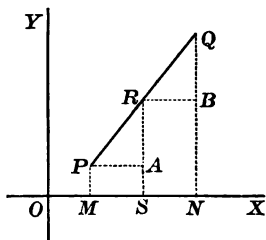


Fig. 4.

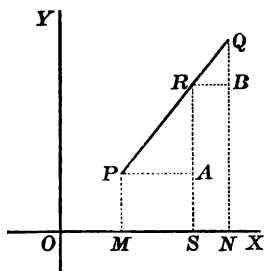


Fig. 5.

Draw  $PM, RS, QN \parallel$  to  $OY$ ; also draw  $PA, RB \parallel$  to  $OX$ .

Then  $\text{rt. } \triangle PRA = \text{rt. } \triangle RQB$  (hypotenuse and one acute angle equal).

Therefore  $PA = RB$ , and  $AR = BQ$ ;

also  $MS = SN$ .

By substitution,  $x - x_1 = x_2 - x$ , and  $y - y_1 = y_2 - y$ ;

whence  $x = \frac{x_1 + x_2}{2}$ ;  $y = \frac{y_1 + y_2}{2}$ . [2]

9. To divide the line joining two given points into two parts having a given ratio  $m : n$ .

Let  $P$  and  $Q$  (Fig. 5) be the given points  $(x_1, y_1)$  and  $(x_2, y_2)$ . Let  $R$  be the required point, such that  $PR : RQ = m : n$ , and let  $x$  and  $y$  denote the co-ordinates of  $R$ .

Complete the figure by drawing lines as in Fig. 4.

The rt.  $\triangle PRA$  and  $RQB$ , being mutually equiangular, are similar; therefore

$$\frac{PA}{RB} = \frac{PR}{RQ} = \frac{m}{n}, \quad \text{and} \quad \frac{AR}{BQ} = \frac{PR}{RQ} = \frac{m}{n}.$$

Substituting for the lines their values, we have

$$\frac{x - x_1}{x_2 - x} = \frac{m}{n}, \quad \text{and} \quad \frac{y - y_1}{y_2 - y} = \frac{m}{n}.$$

Solving these equations for  $x$  and  $y$ , we obtain

$$x = \frac{mx_2 + nx_1}{m + n}, \quad y = \frac{my_2 + ny_1}{m + n}. \quad [3]$$

If  $m = n$ , we have the special case of bisecting a line already considered; and it is easy to see that the values of  $x$  and  $y$  reduce to the forms given in [2].

#### Ex. 4.

What are the co-ordinates of the point

1. Half-way between the points  $(5, 3)$  and  $(7, 9)$ ?
2. Half-way between the points  $(-6, 2)$  and  $(4, -2)$ ?
3. Half-way between the points  $(5, 0)$  and  $(-1, -4)$ ?
4. The vertices of a triangle are  $(2, 3)$ ,  $(4, -5)$ ,  $(-3, -6)$ ; find the middle points of its sides.
5. The middle point of a line is  $(6, 4)$ , and one end of the line is  $(5, 7)$ . What are the co-ordinates of the other end?
6. A line is bisected at the origin; one end of the line is the point  $(-a, b)$ . What are the co-ordinates of the other end?

7. Prove that the middle point of the hypotenuse of a right triangle is equidistant from the three vertices.

8. Prove that the diagonals of a parallelogram mutually bisect each other.

9. Show that the values of  $x$  and  $y$  in [2] hold true when the two given points both lie in the second quadrant.

10. Solve the problem of § 9 when the line  $PQ$  is cut *externally* instead of internally, in the ratio  $m : n$ .

11. What are the co-ordinates of the point which divides the line joining  $(3, -1)$  and  $(10, 6)$  in the ratio  $3 : 4$ ?

12. The line joining  $(2, 3)$  and  $(4, -5)$  is trisected; determine the point of trisection nearest  $(2, 3)$ .

13. A line  $AB$  is produced to a point  $C$ , such that  $BC = \frac{1}{2} AB$ . If  $A$  and  $B$  are the points  $(5, 6)$  and  $(7, 2)$ , what are the co-ordinates of  $C$ ?

14. A line  $AB$  is produced to a point  $C$ , such that  $AB : BC = 4 : 7$ . If  $A$  and  $B$  are the points  $(5, 4)$  and  $(6, -9)$ , what are the co-ordinates of  $C$ ?

15. Three vertices of a parallelogram are  $(1, 2)$ ,  $(-5, -3)$ ,  $(7, -6)$ . What is the fourth vertex?

#### CONSTANTS AND VARIABLES.

10. In Analytic Geometry a line is regarded as a *geometric magnitude traced or generated by a moving point*, — just as we trace on paper what serves to represent a line to the eye by moving the point of a pen or pencil over the paper.

We shall find that great advantages are to be gained by defining a line in this way, but we must be prepared from the outset to make an important distinction in the use of symbols representing lengths. We must distinguish between symbols which denote definite or fixed lengths and those which denote *variable* lengths.

11. A simple example will serve to illustrate this difference. Let  $A$  (Fig. 6) be the point  $(3, 4)$ . Then  $OA = \sqrt{9+16} = 5$ . Now let a point  $P$  describe the line  $OA$  by moving from  $O$  to  $A$ , and let the co-ordinates of  $P$  be denoted by  $x$  and  $y$ ; also let  $z$  denote the length  $OP$  at any position of  $P$ . Then it is clear that the distance  $OP$  or  $z$  will be equal to 0, to begin with, and will increase in value continuously until it becomes equal to 5.

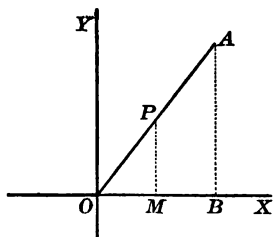


Fig. 6.

Here the word *continuously* deserves special attention. It means that  $P$  must pass successively through every conceivable position on the line  $OA$  from  $O$  to  $A$ ; that, therefore,  $z$  must have in succession every conceivable value between 0 and 5. There will be one position of  $P$  for which  $z$  is equal to 2; there will be another position of  $P$  for which  $z$  is equal to 2.000001; but before reaching this value it must first pass through all values between 2 and 2.000001.

In the same way the co-ordinates of  $P$ , namely,  $x$  and  $y$ , both pass through a continuous series of changes in value unlimited in number, the abscissa  $x$  increasing continuously from 0 to 3, and the ordinate  $y$  from 0 to 4.

We may now divide the lengths considered in this example into two classes:

(1) Lengths supposed to remain constant in value, namely, the co-ordinates of  $A$  and the distance  $OA$ ; (2) lengths supposed to vary continuously in value, namely, the co-ordinates of  $P$ , ( $x$  and  $y$ ), and the distance  $OP$  or  $z$ .



Quantities of the first kind in any problem are called *constant* quantities, or, more briefly, **Constants**.

Quantities of the second kind are called *variable* quantities, or, more briefly, **Variables**.

12. Two variables are often so related that if one of them changes in value, the other also changes in value. The second variable is then said to be a *function* of the first variable. The second variable is also called the *dependent* variable, while the first is called the *independent* variable. Usually the relation between two variables is such that either may be treated as the independent variable, and the other as the dependent variable.

Thus, in § 11, if we suppose  $z$  to change, then both  $x$  and  $y$  will change; the values of  $x$  and  $y$  then will depend upon the value given to  $z$ ; that is,  $x$  and  $y$  will be *functions* of  $z$ . But we may also suppose the value of  $x$ , the abscissa of  $P$ , to change; then it is clear that the values of both  $y$  and  $z$  must also change. In this case we take  $x$  as the independent variable, and values of  $y$  and  $z$  will depend upon the value of  $x$ ; that is,  $y$  and  $z$  will be *functions* of  $x$ .

13. The most concise way to express the relations of the constants and variables which enter into a problem is by means of *algebraic equations*.

The co-ordinates of  $P$  (Fig. 6) throughout its motion are always  $x$  and  $y$ ; and the triangle  $OPM$  is similar to the triangle  $OAB$ . Hence, for any position of  $P$ ,

$$\frac{y}{x} = \frac{4}{3}, \text{ and } z^2 = x^2 + y^2;$$

whence, by solving,

$$y = \frac{4}{3}x, \text{ and } z = \frac{5}{3}x,$$

equations which express the values of  $y$  and  $z$ , respectively, in terms of  $x$  as the independent variable.

(3)

14. In § 11, instead of assuming 3 and 4 as the co-ordinates of  $P$ , we might have employed two letters, as  $a$  and  $b$ , with the understanding that these letters should denote two co-ordinates which remain *constant in value* during the motion of  $P$ . If we choose these letters, and then proceed exactly as in § 13, we obtain for the values of  $y$  and  $z$ ,

$$y = \frac{b}{a}x, \quad z = \frac{\sqrt{a^2 + b^2}}{a}x.$$

15. There is a noteworthy difference between the constants 3 and 4 and the constants  $a$  and  $b$ . The numbers 3 and 4 are unalterable in value; they cannot be supposed to change under any circumstances. The letters  $a$  and  $b$  are constants in this sense only, that they do not change in value when we suppose  $x$  or  $y$  or  $z$  to change in value; in other words, they are not functions of  $x$  or  $y$  or  $z$  in the particular problem under discussion. In all other respects they are free to represent as many different values as we choose to assign to them.

Constants of the first kind (arithmetical numbers) are called *absolute* constants. Constants of the second kind (letters) are called *arbitrary* or *general* constants.

16. By general agreement, variables are represented by the last letters of the alphabet, as  $x, y, z$ ; while constants are most commonly represented by the first letters,  $a, b, c$ , etc., or by the last letters with subscripts added, as  $x_1, y_1, x_2, y_2$ , etc.

#### EX. 5.

1. A point  $P(x, y)$  revolves about the point  $Q(x_1, y_1)$ , keeping always at the distance  $a$  from it. Mention the constants and the variables in this case. What is the total change in the value of each variable?

2. A point  $Q(x, y)$  moves: first parallel to the axis of  $y$ , then parallel to the axis of  $x$ , then equally inclined to the axes. Point out in each case the constants and the variables.

## LOCUS OF AN EQUATION.

17. Let us continue to regard  $x$  and  $y$  as the co-ordinates of a point, and proceed to illustrate the meaning of an algebraic equation containing one or both of these letters.

Take as the first case the equation  $x - 4 = 0$ , whence  $x = 4$ . It is clear that this equation is satisfied by the co-ordinates of every point so situated that its abscissa is equal to 4; therefore it is satisfied by the co-ordinate of every point in the line

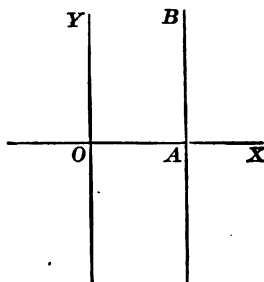


Fig. 7.

$AB$  (Fig. 7), drawn  $\parallel$  to  $OY$ , on the right of  $OY$ , and at the distance 4 from  $OY$ . And it is also clear that this line contains *all* the points whose co-ordinates will satisfy the given equation.

The line  $AB$ , then, may be regarded as the *geometric* representation or meaning of the equation  $x - 4 = 0$ ; and conversely, the equation  $x - 4 = 0$  may be considered to be the *algebraic* representative of this particular line.

In Analytic Geometry the line  $AB$  is called the **locus** of the equation  $x - 4 = 0$ ; conversely, the equation  $x - 4 = 0$  is known as the equation of the line  $AB$ .

The line  $AB$  is to be regarded as extending indefinitely in both directions. If  $AB$  be described by a point  $P$ , moving parallel to the axis of  $y$ , then at all points  $x$  is constant in

value and equal to 4, while  $y$  (which does not appear in the given equation) is a variable, passing through an unlimited number of values, both positive and negative.

18. The equation  $x - y = 0$ , or  $x = y$ , states in algebraic language that the abscissa of the point is always equal to the ordinate.

Values of $x$ .	Values of $y$ .
0 . . . . .	0.
1 . . . . .	1.
2 . . . . .	2.
-1 . . . . .	-1.
etc.	etc.

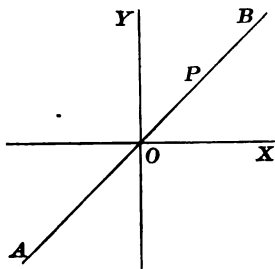


Fig. 8.

If we draw through the origin  $O$  (Fig. 8) a straight line  $AB$ , bisecting the first and third quadrants, then it is easy to see that the given equation is satisfied *by every point* in this line and *by no other points*. If we conceive a point  $P$  to move so that its abscissa shall always be equal to its ordinate, then the point must describe the line  $AB$ . In other words, if the point  $P$  is obliged to move so that its co-ordinates (which of course are variables) shall always satisfy the condition expressed by the equation  $x - y = 0$ ; then the motion of  $P$  is confined to the line  $AB$ .

The line  $AB$  is the locus of the equation  $x - y = 0$ , and this equation represents the line  $AB$ .

19. The equation  $2x + y - 3 = 0$  is satisfied by an unlimited number of values of  $x$  and  $y$ . We may find as many of them as we please by assuming values for one of the variables, and computing the corresponding values of the other.

If we assume for  $x$  the values given below, we easily find for  $y$  the corresponding values given in the next column.

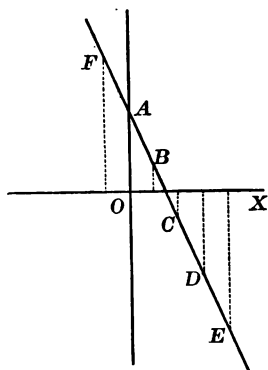


Fig. 9.

Values of $x$ .	Values of $y$ .
0 . . . . .	3.
1 . . . . .	1.
2 . . . . .	-1.
3 . . . . .	-3.
4 . . . . .	-5.
-1 . . . . .	5.
-2 . . . . .	7.
-3 . . . . .	9.
-4 . . . . .	11.
etc.	etc.

Plotting these points (as shown in Fig. 9), we obtain a series of points so-placed that their co-ordinates all satisfy the given equation. By assuming for  $x$  values between 0 and 1, 1 and 2, etc., we might in the same way obtain as many points as we please between  $A$  and  $B$ ,  $B$  and  $C$ , etc. In this case, however, the points all lie in a *straight* line (as will be shown later); so that if any *two* points are found, the straight line drawn through them will include *all* the points whose co-ordinates satisfy the given equation. Now imagine that a point  $P$ , the co-ordinates of which are denoted by  $x$  and  $y$ , is required to move in such a way that the values of  $x$  and  $y$  shall always satisfy the equation  $2x + y - 3 = 0$ ; then  $P$  *must* describe the line  $AB$ , and cannot describe any other line.

The line  $AB$  is the locus of the equation  $2x + y - 3 = 0$ .

**20.** Thus far we have taken equations of the first degree. Let us now consider the equation  $x^2 - y^2 = 0$ . By solving for  $y$ , we obtain  $y = \pm x$ . Hence for every value of  $x$  there are

two values of  $y$ , both equal numerically to  $x$ , but having unlike signs. Thus, for assumed values of  $x$ , we have corresponding values of  $y$  given below :

Values of $x$ .	Values of $y$ .
0 . . . . .	0.
1 . . . . .	1, -1.
2 . . . . .	2, -2.
3 . . . . .	3, -3.
-1 . . . . .	-1, 1.
-2 . . . . .	-2, 2.
-3 . . . . .	-3, 3.

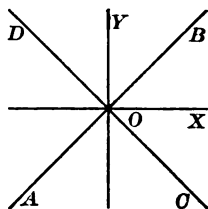


Fig. 10.

By plotting a few points, and comparing this case with the example in § 19, it becomes evident that the locus of the equation consists of *two* lines,  $AB$ ,  $CD$  (Fig. 10), drawn through the origin so as to bisect the four quadrants.

21. There is another way of looking at this case. The equation  $x^2 - y^2 = 0$ , by factoring, may be written  $(x - y)(x + y) = 0$ . Now the equation is satisfied if *either* factor = 0; hence, it is satisfied if  $x - y = 0$ , and also if  $x + y = 0$ . We know (see § 19) that the locus of the equation  $x - y = 0$  is the line  $AB$  (Fig. 8). And the locus of the equation  $x + y = 0$  (or  $x = -y$ ) is evidently the line  $CD$ , since every point in it is so placed that the two co-ordinates are equal numerically but unlike in sign. Therefore the original equation  $x^2 - y^2 = 0$  is represented by the pair of lines  $AB$  and  $CD$  (Fig. 10).

22. Let us next consider the equation  $x^2 + y^2 = 25$ . Solving for  $y$ , we obtain  $y = \pm \sqrt{25 - x^2}$ . When  $x < 5$  there are two values of  $y$  equal numerically but unlike in sign. When  $x = 5$ ,  $y = 0$ . When  $x > 5$  the values of  $y$  are imaginary; this last result means that there is no point with an abscissa greater than 5 whose co-ordinates will satisfy the given equation.

By assigning values of  $x$  differing by unity, we obtain the following sets of values of  $x$  and  $y$ ; and by plotting the points, and then drawing through them a continuous curve, we obtain the curve shown in Fig. 11.

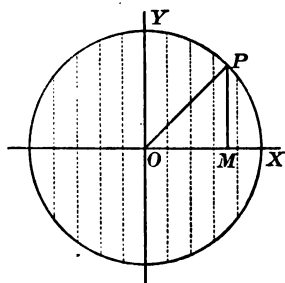


Fig. 11.

Values of $x$ .	Values of $y$ .
0 . . . . .	$\pm 5$ .
1 . . . . .	$\pm 4.9$ .
2 . . . . .	$\pm 4.6$ .
3 . . . . .	$\pm 4$ .
4 . . . . .	$\pm 3$ .
5 . . . . .	0.
-1 . . . . .	$\pm 4.9$ .
-2 . . . . .	$\pm 4.6$ .
-3 . . . . .	$\pm 4$ .
-4 . . . . .	$\pm 3$ .
-5 . . . . .	0.

In this case, however, the locus may be found as follows :

Let  $P$  (Fig. 11) be any point so placed that its co-ordinates,  $x = OM$ ,  $y = MP$ , satisfy the equation  $x^2 + y^2 = 25$ . Join  $OP$ ; then  $x^2 + y^2 = \overline{OP}^2$ ; therefore  $OP = 5$ . Hence, if  $P$  is anywhere in the *circle* described with  $O$  as centre and 5 for radius, its co-ordinates will satisfy the given equation; and if  $P$  is *not* in this circle, its co-ordinates will *not* satisfy the equation. This circle, then, is the locus of the equation.

23. The points whose co-ordinates satisfy the equation  $y^2 = 4x$  lie neither in a straight line nor in a circle. Nevertheless, they do all lie in a certain line, which is, therefore, completely determined by the equation. To construct this line, we first find a number of points which satisfy the equation (the closer the points to one another, the better) and then draw, freehand or with the aid of tracing curves, a continuous curve through the points.

The co-ordinates of a number of such points are given in

the table below. It is evident that for each positive value of  $x$  there are two values of  $y$ , equal numerically but unlike in sign. If we assume a negative value for  $x$ , then the value of  $y$  is imaginary; this result means that there are no points to the left of the axis of  $y$  which will satisfy the given equation.

Values of $x$ .	Values of $y$ .
0 . . . . .	0.
1 . . . . .	$\pm 2$ .
2 . . . . .	$\pm 2.83$ .
3 . . . . .	$\pm 3.46$ .
4 . . . . .	$\pm 4$ .
5 . . . . .	$\pm 4.47$ .
6 . . . . .	$\pm 4.90$ .
7 . . . . .	$\pm 5.29$ .
8 . . . . .	$\pm 5.66$ .
9 . . . . .	$\pm 6$ .
-1 . . . . .	imaginary.

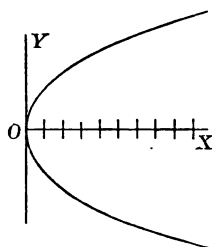


Fig. 12.

In Fig. 12 the several points obtained are plotted, and a smooth curve is then drawn through them. It passes through the origin, is placed symmetrically on both sides of the axis of  $x$ , lies wholly on the right of the axis of  $y$ , and extends towards the right without limit. It is the locus of the given equation, and is a curve called the **Parabola**.

24. After a study of the foregoing examples, we may lay down the following general principles, which form the foundation of the science of Analytic Geometry :

I. Every algebraic equation involving  $x$  and  $y$  is satisfied by an unlimited number of sets of values of  $x$  and  $y$ ; in other words,  $x$  and  $y$  may be treated as *variables*, or quantities varying continuously, yet always so related that their values constantly satisfy the equation.



II. The letters  $x$  and  $y$  may also be regarded as representing the co-ordinates of a point. This point is not fixed in position, because  $x$  and  $y$  are variables; but it cannot be placed at random, because  $x$  and  $y$  can have only such values as will satisfy the equation; now since these values are continuous, the point may be conceived to *move continuously*, and will therefore describe a definite line, or group of lines.

The line, or group of lines, described by a point moving so that its co-ordinates always satisfy the equation is called the **Locus of the Equation**; conversely, the equation satisfied by the co-ordinates of every point in a certain line is called the **Equation of the Line**.

An *equation*, therefore, containing the variables  $x$  and  $y$  is the algebraic representation of a line.

In *Analytic Geometry* the loci considered are represented by their equations, and the investigation of their properties is carried on by means of these equations.

**Ex. 6.**

Determine and construct the loci of the following equations (the locus in each case being either a straight line or a circle):

1.  $x - 6 = 0$ .

9.  $9x^2 - 25 = 0$ .

2.  $x + 5 = 0$ .

10.  $4x^2 - y^2 = 0$ .

3.  $y = -7$ .

11.  $x^2 - 16y^2 = 0$ .

4.  $x = 0$ .

12.  $x^2 + y^2 = 36$ .

5.  $y = 0$ .

13.  $x^2 + y^2 - 1 = 0$ .

6.  $x + y = 0$ .

14.  $x(y + 5) = 0$ .

7.  $x - 2y = 0$ .

15.  $(x - 2)(x - 3) = 0$ .

8.  $2x + 3y + 10 = 0$ .

16.  $(y - 4)(y + 1) = 0$ .

17. What is the geometric meaning of the equation  $5x^2 - 17x - 12 = 0$ ?

HINT. Resolve the equation into two binomial factors.

18. What is the geometric meaning of the equation  $y^2 + 3y = 0$ ?

19. What two lines form the locus of the equation  $xy + 4x = 0$ ?

20. Is the point  $(2, -5)$  situated in the locus of the equation  $4x - 3y - 22 = 0$ ?

HINT. See if the co-ordinates of the point satisfy the equation.

21. Is the point  $(4, -6)$  in the locus of the equation  $y^2 = 9x$ ?

22. Is the point  $(-1, -1)$  in the locus of the equation  $16x^2 + 9y^2 + 15x - 6y - 18 = 0$ ?

23. Does the locus of the equation  $x^2 + y^2 = 100$  pass through the point  $(-6, 8)$ ?

24. Which of the loci represented by the following equations pass through the origin?

(1)  $3x + 2 = 0$ .

(5)  $3x = 2y$ .

(2)  $3x - 11y + 7 = 0$ .

(6)  $3x - 11y = 0$ .

(3)  $x^2 - 16y^2 - 10 = 0$ .

(7)  $x^2 - 16y^2 = 0$ .

(4)  $ax + by + c = 0$ .

(8)  $ax + by = 0$ .

25. The abscissa of a point in the locus of the equation  $3x - 4y - 7 = 0$  is 9; what is the value of the ordinate?

Ans. 5.

26. Determine that point in the locus of  $y^2 - 4x = 0$  for which the ordinate  $= -6$ .

Ans. The point  $(9, -6)$ .

27. Determine the point where the line represented by the equation  $7x + y - 14 = 0$  cuts the axis of  $x$ .

Ans. The point  $(2, 0)$ .

## INTERSECTIONS OF LOCI.

25. The term *Curve*, as used in Analytic Geometry, means any geometric locus, including the straight line as well as lines commonly called curves.

The *Intercepts* of a curve on the axes are the distances from the origin to the points where the curve cuts the axes.

26. *To find the intercepts of a curve, having given its equation.*

The intercept of a curve on the axis of  $x$  is the abscissa of the point where the curve cuts the axis of  $x$ . The ordinate of this point  $= 0$ . Therefore, to find this intercept, put  $y = 0$  in the given equation of the curve, and then solve the equation for  $x$ ; the resulting real values of  $x$  will be the intercepts required.

If the equation is of a higher degree than the first, there will in general be more than one real value of  $x$ ; and the curve will intersect the axis of  $x$  in as many points as there are *real* values of  $x$ .

To an imaginary value of  $x$  there corresponds no intercept; but it is sometimes convenient to speak of such a value as an imaginary intercept.

Similarly, to find the intercepts on the axis of  $y$ , put  $x = 0$  in the given equation, and then solve it for  $y$ ; the resulting real values of  $y$  will be the intercepts required.

27. *To find the points of intersection of two curves, having given their equations.*

Since the points of intersection lie in both curves, their co-ordinates must satisfy both equations. Therefore, to find their co-ordinates, solve the two equations, regarding the variables  $x$  and  $y$  as unknown quantities.

If the equations are both of the first degree, there will be

only *one* pair of values of  $x$  and  $y$ , and one point of intersection.

If the equations are, one or both of them, of higher degree than the first, there may be several pairs of values of  $x$  and  $y$ ; in this case there will be as many points of intersection as there are pairs of *real* values of  $x$  and  $y$ .

If imaginary values of either  $x$  or  $y$  are obtained, there are no corresponding points of intersection.

**28.** *If a curve pass through the origin, its equation, reduced to its simplest form, cannot have a constant term; that is, cannot have a term free from both  $x$  and  $y$ .*

Since in this case the point  $(0, 0)$  is a point of the curve, its equation must be satisfied by the values  $x=0$ , and  $y=0$ . But it is obvious that these values cannot satisfy the equation if, after reduction to its simplest form, it still contains a constant term. Therefore the equation cannot have a constant term.

**29.** *If an equation has no constant term, its locus must pass through the origin.*

For, the values  $x=0$ ,  $y=0$  must evidently satisfy the equation, and therefore the point  $(0, 0)$  must be a point of the locus.

#### Ex. 7.

Find the intercepts of the following curves :

1.  $4x + 3y - 48 = 0$ .

8.  $x - 3 = 0$ .

2.  $5y - 3x - 30 = 0$ .

9.  $x^2 - 9 = 0$ .

3.  $x^2 + y^2 = 16$ .

10.  $x^2 - y^2 = 0$ .

4.  $9x^2 + 4y^2 = 16$ .

11.  $y^2 = 4x$ .

5.  $9x^2 - 4y^2 = 16$ .

12.  $x^2 + y^2 - 4x - 8y = 32$ .

6.  $9x^2 - 4y = 16$ .

13.  $x^2 + y^2 - 4x - 8y = 0$ .

7.  $a^2x^2 + b^2y^2 = a^2b^2$ .

14.  $(x-5)^2 + (y-6)^2 = 20$ .

Find the points of intersection of the following curves :

15.  $3x - 4y + 13 = 0$ ,  $11x + 7y - 104 = 0$ .

16.  $2x + 3y = 7$ ,  $x - y = 1$ .

17.  $x - 7y + 25 = 0$ ,  $x^2 + y^2 = 25$ .

18.  $3x + 4y = 25$ ,  $x^2 + y^2 = 25$ .

19.  $x + y = 8$ ,  $x^2 + y^2 = 34$ .

20.  $2x = y$ ,  $x^2 + y^2 - 10x = 0$ .

21. The equations of the sides of a triangle are  $2x + 9y + 17 = 0$ ,  $7x - y - 38 = 0$ ,  $x - 2y + 2 = 0$ . Find the co-ordinates of its three vertices.

22. The equations of the sides of a triangle are  $5x + 6y = 12$ ,  $3x - 4y = 30$ ,  $x + 5y = 10$ . Find the lengths of its sides.

23. Find the lengths of the sides of a triangle if the equations of the sides are  $x = 0$ ,  $y = 0$ , and  $4x + 3y = 12$ .

24. What are the vertices of the quadrilateral enclosed by the straight lines  $x - a = 0$ ,  $x + a = 0$ ,  $y - b = 0$ ,  $y + b = 0$ ? What kind of a quadrilateral is it?

25. Does the straight line  $5x + 4y = 20$  cut the circle  $x^2 + y^2 = 9$ ?

26. Find the length of that part of the straight line  $3x - 4y = 0$  which is contained within the circle  $x^2 + y^2 = 25$ .

27. Which of the following curves pass through the origin of co-ordinates?

(1)  $7x - 2y + 4 = 0$ .

(4)  $ax + by = 0$ .

(2)  $7x - 2y = 0$ .

(5)  $ax + by + c = 0$ .

(3)  $y^2 - x^2 = 4y$ .

(6)  $x^2 - y + a = a + xy$ .

28. Change the equation  $4x + 2y - 7 = 0$  so that its locus shall pass through the origin.

### CONSTRUCTION OF LOCI.

30. If we know that the locus of a given equation is a straight line, the locus is easily constructed ; it is only necessary to find any two points in it, plot them, and draw a straight line through them with the aid of a ruler.

Likewise, if we know that the locus is a circle, and can find its centre and its radius, the entire locus can then be immediately described with the aid of a pair of compasses.

It will appear later on that the *form* of the given equation enables us at once to tell whether its locus is a straight line or a circle.

If the locus of an equation is neither a straight line nor a circle, then the following method of construction, which is applicable to the locus of any equation without regard to the form of the curve, is usually employed.

#### 31. *To construct the locus of a given equation.*

The steps of the process are as follows :

1. Solve the equation with respect to either  $x$  or  $y$ .
2. Assign values to the other variable, differing not much from one another.
3. Find each corresponding value of the first variable.
4. Draw two axes, choose a suitable scale of lengths, and plot the points whose co-ordinates have been obtained.
5. Draw a continuous curve through these points.

DISCUSSION. An examination of the equation, as shown in the examples given below, enables us to obtain a good general idea of the shape and size of the curve, its position with respect to the axes, etc. ; in this way it serves as an aid in constructing the curve, and as a means of detecting numerical errors made in computing the co-ordinates of the points. Such an examination is called a *discussion* of the equation.

NOTE 1. This method of constructing a locus is from its nature an *approximate* method. But the nearer the points are to one another, the nearer the curve will approach the exact position of the locus.

NOTE 2. In theory, it is immaterial what scale of lengths is used. In practice, the unit of lengths should be determined by the size of the paper compared with the greatest length to be laid off upon it. Paper sold under the name of "*co-ordinate* paper," ruled in small squares,  $\frac{1}{16}$  of an inch long, on each side, will be found very convenient in practice.

### 32. Construct the locus of the equation

$$9x^2 + 4y^2 - 576 = 0.$$

If we solve for both  $x$  and  $y$ , we obtain the following values :

$$y = \pm \frac{2}{3} \sqrt{64 - x^2}, \quad (1)$$

$$x = \pm \frac{3}{2} \sqrt{144 - y^2}. \quad (2)$$

By assigning to  $x$  values differing by unity, and finding corresponding values of  $y$ , we obtain the results given below. To each value of  $x$ , positive or negative, there correspond two values of  $y$ , equal numerically and unlike in sign. By plotting the corresponding points, and drawing a continuous curve through them, we obtain the closed curve shown in Fig. 13.

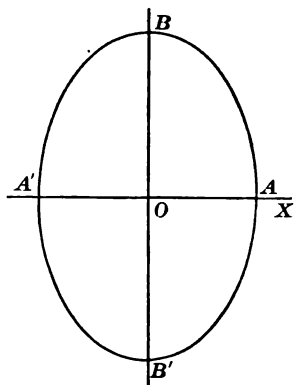


Fig. 13.

Values of $x$ .	Values of $y$ .
0 . . .	$\pm 12$ .
$\pm 1$ . . .	$\pm 11.91$ .
$\pm 2$ . . .	$\pm 11.62$ .
$\pm 3$ . . .	$\pm 11.13$ .
$\pm 4$ . . .	$\pm 10.39$ .
$\pm 5$ . . .	$\pm 9.36$ .
$\pm 6$ . . .	$\pm 7.93$ .
$\pm 7$ . . .	$\pm 5.80$ .
$\pm 8$ . . .	$\pm 0$ .
$\pm 9$ . . .	$\pm$ imaginary.

DISCUSSION. From equations (1) and (2) we see that if  $x = 0$ ,  $y = \pm 12$ , and if  $y = 0$ ,  $x = \pm 8$ ; therefore the intercepts of the curve on the axis of  $x$  are  $+8$  and  $-8$ , and those on the axis of  $y$  are  $+12$  and  $-12$ . These intercepts are the lengths  $OA$ ,  $OA'$ , and  $OB$ ,  $OB'$ , in Fig. 13.

If we assign to  $x$  a numerical value greater than 8, positive or negative, we find by substitution in equation (1) that the corresponding value of  $y$  will be imaginary. This shows that  $OA$  and  $OA'$  are the maximum abscissas of the curve. Similarly, equation (2) shows that the curve has no points with ordinates greater than  $+12$  and  $-12$ .

The greater the numerical value of  $x$ , between the limits 0 and  $+8$  or 0 and  $-8$ , the less the corresponding value of  $y$  numerically; why?

From equation (1) we see that corresponding to each value of  $x$ , between the limits 0 and  $\pm 8$ , there are *two* real values of  $y$ , equal numerically and unlike in sign. Hence, for each value of  $x$  between 0 and  $\pm 8$  there are *two* points of the curve placed equally distant from the axis of  $x$ . Therefore the curve is *symmetrical* with respect to the axis of  $x$ ; in other words, if the portion of the curve above the axis of  $x$  be revolved about this axis through  $180^\circ$ , it will coincide with the portion below the axis. Similarly, it follows from equation (2) that the curve is also symmetrical with respect to the axis of  $y$ . Therefore the entire curve is a closed curve, consisting of four equal quadrantal arcs symmetrically placed about the origin  $O$ . The name of this curve is the **Ellipse**.

33. Construct the locus of the equation

$$4x - y^2 + 16 = 0.$$

Solving for both  $x$  and  $y$ , we obtain

$$y = \pm 2\sqrt{x + 4}, \quad (1)$$

$$x = \frac{y^2 - 16}{4}. \quad (2)$$



We may either assign values to  $x$ , and then compute those of  $y$  by means of (1), or assign values to  $y$ , and compute those of  $x$  by means of (2); the second course is better, because there is less labor in squaring a number than in extracting its square root.

By assigning values to  $y$ , differing by unity from 0 to  $+10$ , and from 0 to  $-10$ , and then proceeding exactly as in the last example, we obtain the series of values given below, and the curve shown in Fig. 14.

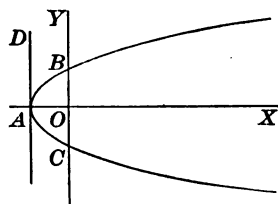


Fig. 14.

Values of $y$ .	Values of $x$ .
0 . . . . .	— 4.
$\pm 1$ . . . . .	— 3.75.
$\pm 2$ . . . . .	— 3.
$\pm 3$ . . . . .	— 1.75.
$\pm 4$ . . . . .	0.
$\pm 5$ . . . . .	2.25.
$\pm 6$ . . . . .	5.
$\pm 7$ . . . . .	8.25.
$\pm 8$ . . . . .	12.
$\pm 9$ . . . . .	16.25.
$\pm 10$ . . . . .	21.

DISCUSSION. An examination of equations (1) and (2) yields the following results, the reasons for which are left as an exercise for the learner :

The intercepts on the axes are :

On the axis of  $x$ ,  $OA = -4$ .

On the axis of  $y$ ,  $OB = +4$ , and  $OC = -4$ .

If we draw through  $A$  the line  $AD \perp$  to  $OX$ , the entire curve lies to the right of  $AD$ .

The curve is situated on both sides of  $OX$ , and is symmetrical with respect to  $OX$ .

The curve extends towards the right without limit.

The curve constantly recedes from  $OX$  as it extends towards the right.

This curve is called a **Parabola**; the point  $A$  is called its **Vertex**, the line  $AX$  its **Axis**.

34. Construct the locus of the equation

$$y = \sin x.$$

If we assume for  $x$  the values  $0^\circ, 10^\circ, 20^\circ, 30^\circ$ , etc., the corresponding values of  $y$  are the *natural* sines of these angles, and are as follows:

Values of $x$ .	Values of $y$ .	Values of $x$ .	Values of $y$ .
$0^\circ$ . . . . .	0.	$50^\circ$ . . . . .	0.77.
$10^\circ$ . . . . .	0.17.	$60^\circ$ . . . . .	0.87.
$20^\circ$ . . . . .	0.34.	$70^\circ$ . . . . .	0.94.
$30^\circ$ . . . . .	0.50.	$80^\circ$ . . . . .	0.98.
$40^\circ$ . . . . .	0.64.	$90^\circ$ . . . . .	1.

If we continue the values of  $x$  from  $90^\circ$  to  $180^\circ$ , the above values of  $y$  repeat themselves in the inverse order (*e.g.*, if  $x = 100^\circ$ ,  $y = 0.98$ , etc.); from  $180^\circ$  to  $360^\circ$  the values of  $y$  are numerically the same, and occur in the same order as between  $0^\circ$  and  $180^\circ$ , but are negative.

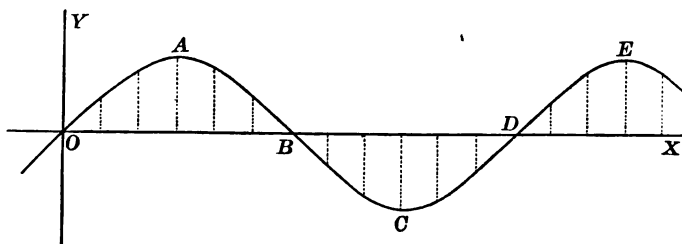


Fig. 15.

In order to express both  $x$  and  $y$  in terms of a common linear unit, we ought, in strictness, to use the circular measure of an angle in which the linear unit represents an angle

of  $57.3^\circ$ , very nearly (see § 5). But it is more convenient, and serves our present purpose equally well, to assume that an angle of  $60^\circ =$  the linear unit. This assumption is made in Fig. 15, where the curve is drawn with one centimeter as the linear unit.

DISCUSSION. The curve passes through the origin, and cuts the axis of  $x$  at points separated by intervals of  $180^\circ$ . Since an angle may have any magnitude, positive or negative, the curve extends on both sides of the origin without limit. The maximum value of the ordinate is alternately  $+1$  and  $-1$ : the former value corresponds to the angle  $90^\circ$ , and repeats itself at intervals of  $360^\circ$ ; the latter value corresponds to the angle  $270^\circ$ , and repeats itself at intervals of  $360^\circ$ . The curve has the form of a wave, and is called the **Sinusoid**.

### Ex. 8.

Construct the loci of the following equations :

- |                                    |                             |
|------------------------------------|-----------------------------|
| 1. $3x - y - 2 = 0$ .              | 13. $y^2 - 1 = 0$ .         |
| 2. $y = 2x$ .                      | 14. $y = x^3$ .             |
| 3. $x^2 = y^2$ .                   | 15. $xy = 12$ .             |
| 4. $x^2 + y^2 = 100$ .             | 16. $x = \sin y$ .          |
| 5. $x^2 - y^2 = 25$ .              | 17. $y = 2 \sin x$ .        |
| 6. $4x^2 - y^2 = 0$ .              | 18. $y = \sin 2x$ .         |
| 7. $4x^2 + 9y^2 = 144$ .           | 19. $y = \cos x$ .          |
| 8. $y^2 - 16x = 0$ .               | 20. $y = \tan x$ .          |
| 9. $y^2 + 16x = 0$ .               | 21. $y = \cot x$ .          |
| 10. $x^2 - 2x - 10y - 5 = 0$ .     | 22. $y = \sec x$ .          |
| 11. $y^2 - 2y - 10x = 0$ .         | 23. $y = \csc x$ .          |
| 12. $(x - 3)^2 + (y - 2)^2 = 25$ . | 24. $y = \sin x + \cos x$ . |

### EQUATION OF A CURVE.

35. From what precedes, we may conclude that every equation involving  $x$  and  $y$  as variables represents a definite line (or group of lines) known as the *locus* of the equation. Regarded from this point of view, an equation is the statement in algebraic language of a *geometric condition* which must always be satisfied by a point  $(x, y)$ , as we imagine it to move in the plane of the axes. For example, the equation  $x = 2y$  states the condition that the point must so move that its abscissa shall always be equal to twice its ordinate; the equation  $x^2 + y^2 = 4$  states the condition that the point must so move that the sum of the squares of its co-ordinates shall always be equal to 4; etc.

Conversely, every geometric condition that a point is required to satisfy must confine the point to a definite line as its locus, and must lead to an equation that is always satisfied by the co-ordinates of the point.

Hence arises a new problem, and one usually of greater difficulty than any thus far considered, namely:

*Given the geometric condition to be satisfied by a point, to find the equation of its locus.*

The great importance of this problem lies in the fact that in the practical applications of Analytic Geometry the law of a moving point is commonly the one thing known to start with, so that the first step must consist in finding the equation of its locus.

#### Ex. 9.

1. A point moves so that it is always three times as far from the axis of  $x$  as from the axis of  $y$ . What is the equation of its locus?

2. What is the equation of the locus of a point which moves so that its abscissa is always equal to  $+6$ ?  $-6$ ?  $0$ ?

3. What is the equation of the locus of a point which moves so that its ordinate is always equal to  $+4$ ?  $-1$ ?  $0$ ?

4. A point so moves that its distance from the straight line  $x = 3$  is always numerically equal to 2. What is the equation of its locus?

5. A point so moves that its distance from the straight line  $y = 5$  is always numerically equal to 3. Find the equation of its locus. Construct the locus.

6. A point moves so that its distance from the straight line  $x + 4 = 0$  is always numerically equal to 5. Find the equation of its locus. Construct the locus.

7. What is the equation of the locus of a point equidistant

(1) from the parallels  $x = 0$  and  $x = -6$ ?

(2) from the parallels  $y = 7$  and  $y = -3$ ?

8. What is the equation of the locus of a point always equidistant from the origin and the point  $(6, 0)$ ?

Find the equation of the locus of a point

9. Equidistant from the points  $(4, 0)$  and  $(-2, 0)$ .

10. Equidistant from the points  $(0, -5)$  and  $(0, 9)$ .

11. Equidistant from the points  $(3, 4)$  and  $(5, -2)$ .

12. Equidistant from the points  $(5, 0)$  and  $(0, 5)$ .

13. A point moves so that its distance from the origin is always equal to 10. Find the equation of its locus.

14. A point moves so that its distance from the point  $(4, -3)$  is always equal to 5. Find the equation of its locus, and construct it. What kind of curve is it? Does it pass through the origin? Why?

15. What is the equation of the locus of a point whose distance from the point  $(-4, -7)$  is always equal to 8?

16. About the origin of co-ordinates as centre, with a radius equal to 5, a circle is described. A point outside this circle so moves that its distance from the circumference of the circle is always equal to 4. What is the equation of its locus?

17. A high rock  $A$ , rising out of the water, is 3 miles from a perfectly straight shore  $BC$ . A vessel so moves that its distance from the rock is always the same as its distance from the shore. What is the equation of its locus?

18. A point  $A$  is situated at the distance 6 from the line  $BC$ . A moving point  $P$  is always equidistant from  $A$  and  $BC$ . Find the equation of its locus.

19. A point moves so that its distance from the axis of  $x$  is half its distance from the origin; find the equation of its locus.

20. A point moves so that the sum of the squares of its distances from the two fixed points  $(a, 0)$  and  $(-a, 0)$  is the constant  $2k^2$ ; find the equation of its locus.

21. A point moves so that the difference of the squares of its distances from  $(a, 0)$  and  $(-a, 0)$  is the constant  $k^2$ ; find the equation of its locus.

#### Ex. 10. (Review.)

1. If we should plot all possible points for which  $x = -5$ , how would they be situated?

2. Construct the point  $(x, y)$  if  $x = 2$  and

$$(1) y = 4x - 3, \quad (2) 3x - 2y = 8.$$

3. The vertices of a rectangle are the points  $(a, b)$ ,  $(-a, b)$ ,  $(-a, -b)$ , and  $(a, -b)$ . Find the lengths of its sides, the lengths of its diagonals, and show that the vertices are equidistant from the origin.

4. What does equation [1], p. 6, for the distance between two points, become when one of the points is the origin?

5. Express by an equation that the distance of the point  $(x, y)$  from the point  $(4, 6)$  is equal to 8.

6. Express that the point  $(x, y)$  is equidistant from the points  $(2, 3)$  and  $(4, 5)$ .

7. Find the point equidistant from the points  $(2, 3)$ ,  $(4, 5)$ , and  $(6, 1)$ . What is the common distance?

8. Prove that the diagonals of a rectangle are equal.

9. Prove that the diagonals of a parallelogram mutually bisect each other.

10. The co-ordinates of three vertices of a parallelogram are known:  $(5, 3)$ ,  $(7, 10)$ ,  $(13, 9)$ . What are the co-ordinates of the remaining vertex?

11. The co-ordinates of the vertices of a triangle are  $(3, 5)$ ,  $(7, -9)$ ,  $(2, -4)$ . Find the co-ordinates of the middle points of its sides.

12. The centre of gravity of a triangle is situated on the line joining any vertex to the middle point of the opposite side, at the point of trisection nearest that side. Find the centre of gravity of the triangle whose vertices are the points  $(2, 3)$ ,  $(4, -5)$ ,  $(-3, -6)$ .

13. The vertices of a triangle are  $(5, -3)$ ,  $(7, 9)$ ,  $(-9, 6)$ . Find the distance from its centre of gravity to the origin.

14. The vertices of a quadrilateral are  $(0, 0)$ ,  $(5, 0)$ ,  $(9, 11)$ ,  $(0, 3)$ . Find the co-ordinates of the intersection of the two straight lines which join the middle points of its opposite sides.

15. Prove that the two straight lines which join the middle points of the opposite sides of any quadrilateral mutually bisect each other.

16. A line is divided into three equal parts. One end of the line is the point  $(3, 8)$ ; the adjacent point of division is  $(4, 13)$ . What are the co-ordinates of the other end?

17. The line joining the points  $(x_1, y_1)$  and  $(x_2, y_2)$  is divided into four equal parts. Find the co-ordinates of the points of division.

18. Explain and illustrate the relation which exists between an equation and its locus.

19. Construct the two lines which form the locus of the equation  $x^2 - 7x = 0$ .

20. Is the point  $(2, -5)$  in the locus of the equation  $4x^2 - 9y^2 = 36$ ?

21. The ordinate of a certain point in the locus of the equation  $x^2 + y^2 + 20x - 70 = 0$  is 1. What is the abscissa of this point?

22. Find the intercepts of the curve  $x^2 + y^2 - 5x - 7y + 6 = 0$ .

Find the points common to the curves :

23.  $x^2 + y^2 = 100$ , and  $y^2 - \frac{9x}{2} = 0$ .

24.  $x^2 + y^2 = 5a^2$ , and  $x^2 = 4ay$ .

25.  $b^2x^2 + a^2y^2 = a^2b^2$ , and  $x^2 + y^2 = a^2$ .

26. Find the lengths of the sides of a triangle, if its vertices are  $(6, 0)$ ,  $(0, -8)$ ,  $(-4, -2)$ .

27. A point moves so that it is always six times as far from one of two fixed perpendicular lines as from the other. Find the equation of its locus.

28. A point so moves that its distance from the fixed point  $A$  is always double its distance from the fixed line  $AB$ . Find the equation of its locus.

29. A fixed point is at the distance  $a$  from a fixed straight line. A point so moves that its distance from the fixed point is always twice its distance from the fixed line. Find the equation of its locus.



## CHAPTER II.

### THE STRAIGHT LINE.

#### EQUATIONS OF THE STRAIGHT LINE.

36. *Notation.* Throughout this chapter, and generally in equations of straight lines,

$a$  = the intercept on the axis of  $x$ .

$b$  = the intercept on the axis of  $y$ .

$\gamma$  = the angle between the axis of  $x$  and the line.

$m = \tan \gamma$ .

$p$  = the perpendicular from the origin to the line.

$\alpha$  = the angle between the axis of  $x$  and  $p$ .

These six quantities are *general constants*;  $a$ ,  $b$ , and  $m$  may have any values from  $-\infty$  to  $+\infty$ ;  $p$ , any value from 0 to  $+\infty$ ;  $\gamma$ , any value from  $0^\circ$  to  $180^\circ$ ;  $\alpha$ , any value from  $0^\circ$  to  $360^\circ$ .

The constant  $m$  is often called the **Slope** of the line; its value determines the direction of the line.

In order to determine a straight line, *two* geometric conditions must be given.

37. *To find the equation of a straight line passing through two given points  $(x_1, y_1)$  and  $(x_2, y_2)$ .*

Let  $A$  (Fig. 16) be the point  $(x_1, y_1)$ ,  $B$  the point  $(x_2, y_2)$ ; and let  $P$  be any other point of the line drawn through  $A$  and  $B$ ,  $x$  and  $y$  its co-ordinates. Draw  $AC$ ,  $BD$ ,  $PM$ ,  $\parallel$  to  $OY$ , and  $AEF \parallel$  to  $OX$ .

The triangles  $APF$ ,  $ABE$  are similar ; therefore

$$\frac{PF}{AF} = \frac{BE}{AE}.$$

Now  $PF = y - y_1$ ,  $AF = x - x_1$ ,  $BE = y_2 - y_1$ ,  $AE = x_2 - x_1$ .

Therefore 
$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}. \quad [4]$$

This is the equation required.

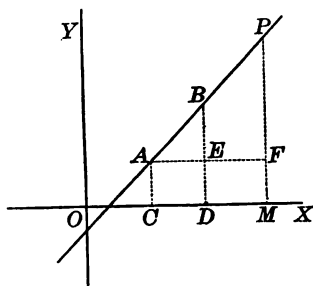


Fig. 16.

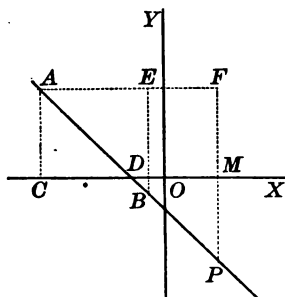


Fig. 17.

It is evident that the angle  $FAP = \gamma$ . Therefore each side of equation [4] is equal to  $\tan \gamma$  or  $m$ . The first side contains the two variables  $x$  and  $y$ , and the equation tells us that they must vary in such a way that the fraction  $\frac{y - y_1}{x - x_1}$  shall remain constant in value, and always equal to  $m$ .

NOTE. In Fig. 16 the points  $A$ ,  $B$ , and  $P$  are assumed in the first quadrant in order to avoid negative quantities. But the reasoning will lead to equation [4] whatever be the positions of these points. In Fig. 17 the points are in different quadrants. The triangles  $APF$ ,  $ABE$  are to be constructed as shown in the figure. They are similar; and by taking proper account of the algebraic signs of the quantities, we arrive at equation [4], as before. The learner should study this case with care, and should study other cases devised by himself, till he is convinced that equation [4] is perfectly general.

38. To find the equation of a straight line, given one point  $(x_1, y_1)$  in the line and the slope  $m$ .

Let the figure be constructed like Fig. 16, omitting the point  $B$  and the line  $BED$ . Then it is evident that

$$m = \frac{PF}{AF} = \frac{y - y_1}{x - x_1};$$

whence,

$$y - y_1 = m(x - x_1). \quad [5]$$

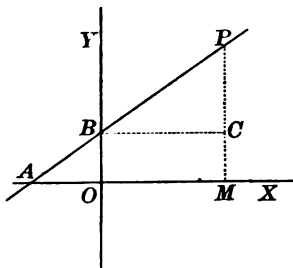


Fig. 18.

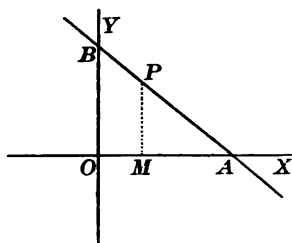


Fig. 19.

39. To find the equation of a straight line, given the intercept  $b$  and the angle  $\gamma$ .

Let the line cut the axes in the points  $A$  and  $B$  (Fig. 18). Let  $P$  be any point  $(x, y)$  in the line. Draw  $PM \parallel$  to  $OY$ , and  $BC \parallel$  to  $OX$ .

Then  $OB = b$ ,  $PBC = \gamma$ ,  $BC = x$ ,  $PC = y - b$ ;

therefore 
$$m = \frac{y - b}{x};$$

whence

$$y = mx + b. \quad [6]$$

40. To find the equation of a straight line, given its intercepts  $a$  and  $b$ .

Let the line cut the axes in the points  $A$  and  $B$  (Fig. 19), and let  $P$  be any other point  $(x, y)$  in the line. Then  $OA = a$ ,

$OB = b$ . Draw  $PM \perp$  to  $OX$ . The triangles  $PMA$ ,  $BOA$  are similar; therefore

$$\frac{PM}{BO} = \frac{MA}{OA} = \frac{OA - OM}{OA},$$

or

$$\frac{y}{b} = \frac{a - x}{a} = 1 - \frac{x}{a};$$

whence

$$\frac{x}{a} + \frac{y}{b} = 1. \quad [7]$$

This is called the **Symmetrical Equation** of the straight line.

41. To find the equation of a straight line, given its distance  $p$  from the origin, and the angle  $\alpha$ .

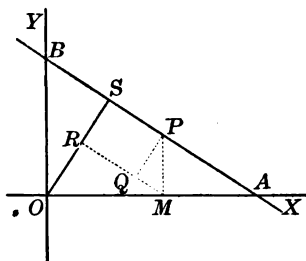


Fig. 20.

Let  $AB$  (Fig. 20) be the line,  $P$  any point in it. Draw  $OS \perp$  to  $AB$ , meeting  $AB$  in  $S$ ;  $PM \perp$  to  $OX$ ;  $MR \parallel$  to  $AB$ , meeting  $OS$  in  $R$ ; and  $PQ \perp$  to  $AB$ .

Then  $p = OS = OR + QP$ ,  $\alpha = XO S = PMQ$ .

By Trigonometry

$$OR = OM \cos \alpha = x \cos \alpha,$$

$$QP = PM \sin \alpha = y \sin \alpha.$$

Therefore  $OR + QP = p = x \cos \alpha + y \sin \alpha$ .

Or  $x \cos \alpha + y \sin \alpha = p. \quad [8]$

This is called the **Normal Equation** of the straight line.

The coefficients  $\cos \alpha$  and  $\sin \alpha$  determine the direction of the line, and  $p$  its distance from the origin.

NOTE. Observe that all the equations of the straight line which have been obtained are of the *first* degree. Their differences in form are due to the constants which enter them. The form of each, and the signification of its constants, should be thoroughly fixed in mind.

## EX. 11.

Find the equation of the straight line passing through the two points:

- |                           |                                  |
|---------------------------|----------------------------------|
| 1. (2, 3) and (4, 5).     | 7. (2, 5) and (0, 7).            |
| 2. (4, 5) and (7, 11).    | 8. (3, 4) and (0, 0).            |
| 3. (-1, 2) and (3, -2).   | 9. (3, 0) and (0, 0).            |
| 4. (-2, -2) and (-3, -3). | 10. (3, 4) and (-2, 4).          |
| 5. (4, 0) and (2, 3).     | 11. (2, 5) and (-2, -5).         |
| 6. (0, 2) and (-3, 0).    | 12. ( $m, n$ ) and ( $-m, -n$ ). |

Find the equation of a straight line, given :

- |   |                                    |
|---|------------------------------------|
| 13. (4, 1) and $\gamma = 45^\circ$ .    | 29. $b = -4, \gamma = 120^\circ$ . |
| 14. (2, 7) and $\gamma = 60^\circ$ .    | 30. $b = -4, \gamma = 135^\circ$ . |
| 15. (-3, 11) and $\gamma = 45^\circ$ .  | 31. $b = -4, \gamma = 150^\circ$ . |
| 16. (13, -4) and $\gamma = 150^\circ$ . | 32. $b = -4, \gamma = 180^\circ$ . |
| 17. (3, 0) and $\gamma = 30^\circ$ .    | 33. $a = 4, b = 3$ .               |
| 18. (0, 3) and $\gamma = 135^\circ$ .   | 34. $a = -6, b = 2$ .              |
| 19. (0, 0) and $\gamma = 120^\circ$ .   | 35. $a = -3, b = -3$ .             |
| 20. (2, -3) and $\gamma = 0^\circ$ .    | 36. $a = 5, b = -3$ .              |
| 21. (2, -3) and $\gamma = 90^\circ$ .   | 37. $a = -10, b = 5$ .             |
| 22. $b = 2$ and $\gamma = 45^\circ$ .   | 38. $a = 1, b = -1$ .              |
| 23. $b = 5, \gamma = 45^\circ$ .        | 39. $a = n, b = -n$ .              |
| 24. $b = -4, \gamma = 45^\circ$ .       | 40. $a = n, b = 4n$ .              |
| 25. $b = -4, \gamma = 30^\circ$ .       | 41. $p = 5, a = 45^\circ$ .        |
| 26. $b = -4, \gamma = 0^\circ$ .        | 42. $p = 5, a = 120^\circ$ .       |
| 27. $b = -4, \gamma = 60^\circ$ .       | 43. $p = 5, a = 240^\circ$ .       |
| 28. $b = -4, \gamma = 90^\circ$ .       | 44. $p = 5, a = 300^\circ$ .       |

Write the equations of the sides of a triangle :

45. If its vertices are the points  $(2, 1)$ ,  $(3, -2)$ ,  $(-4, -1)$ .

46. If its vertices are the points  $(2, 3)$ ,  $(4, -5)$ ,  $(-3, -6)$ .

47. Form the equations of the medians of the triangle described in Ex. 46.

48. The vertices of a quadrilateral are  $(0, 0)$ ,  $(1, 5)$ ,  $(7, 0)$ ,  $(4, -9)$ . Form the equations of its sides, and also of its diagonals.

Find the equation of a straight line, given :

49.  $a = 7\frac{1}{2}$ ,  $\gamma = 30^\circ$ .      51.  $p = 6$ ,  $\gamma = 45^\circ$ .

50.  $a = -3$ ,  $(x_1, y_1)$  is  $(2, 5)$ .      52.  $p = 6$ ,  $\gamma = 135^\circ$ .

Reduce the following equations to the symmetrical form, and construct each by its intercepts :

53.  $3x - 2y + 11 = 0$  and  $y = 7x + 1$ .

54.  $3x + 5y - 13 = 0$  and  $4x - y - 2 = 0$ .

55. Reduce  $Ax + By = C$  to the symmetrical form ; also  $y = mx + b$ . What are the values of  $a$  and  $b$  in terms of  $A$ ,  $B$ ,  $C$ , and  $m$  ?

Reduce the following equations to the form  $y = mx + b$ , and construct each by its slope, and intercept on the axis of  $y$  :

56.  $y + 13 = 5x$  and  $y + 19 = 7x$ .

57.  $3x + y + 2 = 0$  and  $2y = 3x + 6$ .

58. Reduce  $Ax + By = C$  to the form  $y = mx + b$  ; also  $\frac{x}{a} + \frac{y}{b} = 1$ . What are the values of  $m$  and  $b$  in terms of  $A$ ,  $B$ ,  $C$ , and  $a$  ?

59. Find the vertices of the triangle whose sides are the lines  $2x + 9y + 17 = 0$ ,  $y = 7x - 38$ ,  $2y - x = 2$ .

60. Find the equation of the straight line passing through the origin and the intersection of the lines  $3x - 2y + 4 = 0$  and  $3x + 4y = 5$ . Also find the distance between these two points.

61. What is the equation of the line passing through  $(x_1, y_1)$ , and equally inclined to the two axes?

62. Find the equations of the diagonals of the parallelogram formed by the lines  $x = a$ ,  $x = b$ ,  $y = c$ ,  $y = d$ .

63. Show that the lines  $y = 2x + 3$ ,  $y = 3x + 4$ ,  $y = 4x + 5$  all pass through one point.

Find the intersection of two of the lines, and then see if its co-ordinates will satisfy the equation of the remaining line.

64. The vertices of a triangle are  $(0, 0)$ ,  $(x_1, 0)$ ,  $(x_2, y_2)$ . Find the equations of its medians, and prove that they meet in one point.

65. What must be the value of  $m$  if the line  $y = mx$  passes through the point  $(1, 4)$ ?

66. The line  $y = mx + 3$  passes through the intersection of the lines  $y = x + 1$  and  $y = 2x + 2$ . Determine the value of  $m$ .

67. Find the value of  $b$  if the line  $y = 6x + b$  passes through the point  $(2, 3)$ .

68. What condition must be satisfied if the points  $(x_1, y_1)$ ,  $(x_2, y_2)$ ,  $(x_3, y_3)$  lie in one straight line?

HINT. Let equation [4] represent the line through  $(x_1, y_1)$  and  $(x_2, y_2)$ ; then  $(x_3, y_3)$  must satisfy it.

69. Discuss equation [5] for the following cases: (i.)  $(x_1, y_1)$  is  $(0, 0)$ , (ii.)  $m = 0$ , (iii.)  $m = \infty$ .

70. Discuss equation [6] for the following cases: (i.)  $b = 0$ , (ii.)  $m = 0$ , (iii.)  $m = \infty$ , (iv.)  $m = 0$ , and  $b = 0$ .

71. Discuss equation [7] for the following cases: (i.)  $a = b$ , (ii.)  $a = 0$ , (iii.)  $a = \infty$ , (iv.)  $b = \infty$ .

## GENERAL EQUATION OF THE FIRST DEGREE.

42. If any point  $P(x_1, y_1)$  be connected with the origin  $O$ ; then  $\frac{x_1}{OP} = \cos XOP$ ,  $\frac{y_1}{OP} = \sin XOP$ , and  $OP = \sqrt{x_1^2 + y_1^2}$ .

Hence, if any two real quantities be each divided by the square root of the sum of their squares, the quotients are the cosine and sine of some angle.

43. The locus of every equation of the first degree in  $x$  and  $y$  is a straight line.

Any simple equation in  $x$  and  $y$  can be reduced to the form

$$Ax + By = C, \quad [9]$$

in which  $C$  is positive or zero.

Dividing both members of [9] by  $\sqrt{A^2 + B^2}$ , we obtain

$$\frac{A}{\sqrt{A^2 + B^2}}x + \frac{B}{\sqrt{A^2 + B^2}}y = \frac{C}{\sqrt{A^2 + B^2}}. \quad (1)$$

Now, by § 42, the coefficients of  $x$  and  $y$  in (1) are a set of values of  $\cos a$  and  $\sin a$ , and the second member, being positive, is some value of  $p$  (§ 41). Hence (1) is in the normal form, and its locus is some straight line. Whence the proposition.

COR. 1. To reduce any simple equation to the normal form, put it in the form of [9], and divide both members by the square root of the sum of the squares of the coefficients of  $x$  and  $y$ .

COR. 2. To construct (1), locate the point  $(A, B)$ , connect it with the origin, and on this line lay off  $OS$  equal to the second member of (1); the perpendicular to  $OS$  through  $S$  is the locus of (1), or [9].

44. The locus of an equation of the first degree in  $x$  and  $y$  is called a **Locus of the First Order**.



## Ex. 12.

Reduce the following equations to the normal form, and thus determine  $p$ , or the distance of each locus from the origin :

- |                         |                        |
|-------------------------|------------------------|
| 1. $3x - 2y + 11 = 0$ . | 5. $y + 13 = 5x$ .     |
| 2. $3x + 5y - 13 = 0$ . | 6. $y + 19 = 7x$ .     |
| 3. $4x - y - 2 = 0$ .   | 7. $ex + cy + n = 0$ . |
| 4. $2x + 3y = 7$ .      | 8. $ny + cx - r = 0$ . |

Reduce the following equations to one of the forms [6], [7], [8], and determine by the signs of the constants which of the four quadrants each locus crosses :

- |                             |                                |
|-----------------------------|--------------------------------|
| 9. $y = \frac{1}{3}x - 9$ . | 14. $5x + 4y - 20 = 0$ .       |
| 10. $3x + 2 = 2y$ .         | 15. $y = 6x + 12$ .            |
| 11. $4y = 5x - 1$ .         | 16. $y + 2 = x - 4$ .          |
| 12. $4y = 3x + 24$ .        | 17. $x + \sqrt{3}y + 10 = 0$ . |
| 13. $5x + 3y + 15 = 0$ .    | 18. $x - \sqrt{3}y - 10 = 0$ . |

19. Discuss equation [9] for the following cases :

- |                  |                      |                           |
|------------------|----------------------|---------------------------|
| (i.) $A = 0$ .   | (iv.) $A = \infty$ . | (vii.) $A = B, C = 0$ .   |
| (ii.) $B = 0$ .  | (v.) $A = C = 0$ .   | (viii.) $A = -B, C = 0$ . |
| (iii.) $C = 0$ . | (vi.) $A = B$ .      |                           |

20. Reduce equation [7] to the form of equation [6], and find the value of  $m$  in terms of  $a$  and  $b$ .

21. What value must  $C$  have in order that the line  $4x - 5y = C$  may pass through the origin? Through  $(2, 0)$ ?

22. Determine the values of  $A$ ,  $B$ , and  $C$ , so that the line  $Ax + By = C$  may pass through  $(3, 0)$  and  $(0, -12)$ .

Since the co-ordinates of the given points must satisfy the equation, we have the two relations  $3A = C$  and  $-12B = C$ .

23. From [9] deduce [4] by the method used in No 22.

24. If equations [4] and [9] represent the same line, what are the values of  $A$ ,  $B$ ,  $C$ , in terms of  $x_1$ ,  $y_1$ ,  $x_2$ , and  $y_2$ ?

25. In equation [4] find the values of  $m$  and  $b$  in terms of  $x_1$ ,  $y_1$ ,  $x_2$ ,  $y_2$ .

### ANGLES.

45. To find the angle formed by the lines  $y = mx + b$ , and  $y = m'x + b'$ .

Let  $AB$  and  $CD$  (Fig. 21) represent the two lines respectively, meeting in the point  $P$ .

Let the angle  $APC = \phi$ ; then, by Geometry,  $\phi = \gamma - \gamma'$ . Whence, by Trigonometry,

$$\tan \phi = \frac{m - m'}{1 + mm'}. \quad [10]$$

This equation determines the value of  $\phi$ .

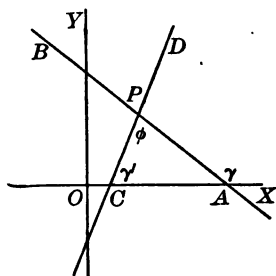


Fig. 21.

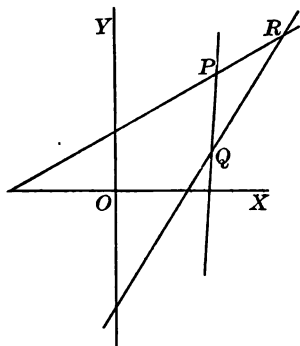


Fig. 22.

COR. 1. If the lines are parallel,  $\tan \phi = 0$ ; hence,  $m = m'$ . Conversely, if  $m = m'$ ,  $\phi = 0$ , and the lines are parallel.

COR. 2. If the lines are perpendicular,  $\tan \phi = \infty$ ; hence,  $1 + mm' = 0$ , or  $m' = -\frac{1}{m}$ . Conversely, if  $1 + mm' = 0$ ,  $\phi = 90^\circ$ , and the lines are perpendicular.

**46.** To find the equations of a straight line passing through the point  $(x_1, y_1)$  and (i.) parallel, (ii.) perpendicular, to the line  $y = mx + b$ .

The slope of the required line is  $m$  in case (i.), and  $-\frac{1}{m}$  in case (ii.); and in both cases the line passes through a given point  $(x_1, y_1)$ .

Therefore (§ 38) the required equations are

$$(1) y - y_1 = m(x - x_1),$$

$$(2) y - y_1 = -\frac{1}{m}(x - x_1).$$

### Ex. 13.

Find the equation of the straight line

1. Passing through  $(3, -7)$ , and  $\parallel$  to the line  $y = 3x - 5$ .
2. Passing through  $(5, 3)$ , and  $\parallel$  to the line  $\frac{1}{3}y - \frac{1}{4}x = 1$ .
3. Passing through  $(0, 0)$ , and  $\parallel$  to the line  $y - 4x = 10$ .
4. Passing through  $(5, 8)$ , and  $\parallel$  to the axis of  $x$ .
5. Passing through  $(5, 8)$ , and  $\parallel$  to the axis of  $y$ .
6. Passing through  $(3, -13)$ , and  $\perp$  to the line  $y = 4x - 7$ .
7. Passing through  $(2, 9)$ , and  $\perp$  to the line  $7y + 23x - 5 = 0$ .
8. Passing through  $(0, 0)$ , and  $\perp$  to the line  $x + 2y = 1$ .
9. Perpendicular to the line  $5x - 7y + 1 = 0$ , and erected at the point whose abscissa = 1.

**47.** To find the equation of a straight line passing through a given point  $(x_1, y_1)$ , and making a given angle  $\phi$  with a given line  $y = mx + b$ .

Let the required equation be

$$y - y_1 = m'(x - x_1),$$

where  $m'$  is not yet determined.

Since the required line may lie either as  $PQ$  or  $PR$  (Fig. 22), we shall have (§ 45),

$$\tan \phi = \frac{m' - m}{1 + mm'} \text{ or } \frac{m - m'}{1 + mm'}.$$

Hence 
$$m' = \frac{m \pm \tan \phi}{1 \mp m \tan \phi},$$

and the required equation is .

$$y - y_1 = \frac{m \pm \tan \phi}{1 \mp m \tan \phi} (x - x_1), \quad [11]$$

and (as Fig. 22 shows) there are in general two straight lines satisfying the given conditions.

#### Ex. 14.

1. Find the angle formed by the lines  $x + 2y + 1 = 0$  and  $x - 3y - 4 = 0$ .

The two slopes are  $-\frac{1}{2}$  and  $\frac{1}{3}$ . If we put  $m = -\frac{1}{2}$ ,  $m' = \frac{1}{3}$ , we obtain  $\tan \phi = -1$ ,  $\phi = 135^\circ$ . If we put  $m = \frac{1}{3}$ ,  $m' = -\frac{1}{2}$ , we get  $\tan \phi = 1$ ,  $\phi = 45^\circ$ . Show that both these results are correct.

Find the tangent of the angle formed by the lines

2.  $3x - 4y = 7$  and  $2x - y = 3$ .

3.  $2x + 3y + 4 = 0$  and  $3x + 4y + 5 = 0$ .

4.  $y - nx = 1$  and  $2(y - 1) = nx$ .

Find the angle formed by the lines

5.  $x + y = 1$  and  $y = x + 4$ .

6.  $y + 3 = 2x$  and  $y + 3x = 2$ .

7.  $2x + 3y + 7 = 0$  and  $3x - 2y + 4 = 0$ .

8.  $6x = 2y + 3$  and  $y - 3x = 10$ .

9.  $x + 3 = 0$  and  $y - \sqrt{3}x + 4 = 0$ .

10. Discuss equation [11] for the cases where  $\phi = 0^\circ$  and  $\phi = 90^\circ$ .

NOTE. The learner should try to solve the next five exercises directly, without using equation [11]; then verify the result by means of [11].

Find the equation of a straight line

11. Passing through the point (3, 5), and making the angle  $45^\circ$  with the line  $2x - 3y + 5 = 0$ .

12. Passing through the point  $(-2, 1)$ , and making the angle  $45^\circ$  with the line  $2y = 6 - 3x$ .

13. Passing through that point of the line  $y = 2x - 1$  for which  $x = 2$ , and making the angle  $30^\circ$  with the same line.

14. Passing through  $(1, 3)$ , and making the angle  $30^\circ$  with the line  $x - 2y + 1 = 0$ .

15. Prove that the lines represented by the equations

$$Ax + By + C = 0, \quad A'x + B'y + C' = 0$$

are parallel if  $AB' = A'B$ ; perpendicular, if  $AA' = -BB'$ .

16. Given the equation  $3x + 4y + 6 = 0$ ; show that the general equations representing (i.) all parallels and (ii.) all perpendiculars to the given line are

$$(i.) \quad 3x + 4y + K = 0.$$

$$(ii.) \quad 4x - 3y + K = 0.$$

17. Deduce the following equations for lines passing through  $(x_1, y_1)$  and (i.) parallel, (ii.) perpendicular, to the line  $y = mx + b$ .

$$(i.) \quad y - mx = y_1 - mx_1.$$

$$(ii.) \quad my + x = my_1 + x_1.$$

18. Write the equations of 3 lines parallel, and 3 lines perpendicular, to the line  $2x + 3y + 1 = 0$ .

19. Among the following lines select parallel lines; perpendicular lines; lines neither parallel nor perpendicular:

- |                          |                             |
|--------------------------|-----------------------------|
| (i.) $2x + 3y - 1 = 0$ . | (v.) $x - y = 2$ .          |
| (ii.) $3x - 2y = 20$ .   | (vi.) $5(x + y) - 11 = 0$ . |
| (iii.) $4x + 6y = 0$ .   | (vii.) $x = 8$ .            |
| (iv.) $12x = 8y + 7$ .   | (viii.) $y + 10 = 0$ .      |

20. Prove that the angle  $\phi$ , between the lines

$$Ax + By + C = 0 \text{ and } A'x + B'y + C' = 0,$$

is determined by the equation

$$\tan \phi = \frac{A'B - AB'}{AA' + BB'}.$$

21. From the preceding equation deduce the conditions of parallel lines and perpendicular lines given in No. 15.

Find the equation of a straight line

22. Parallel to  $2x + 3y + 6 = 0$ , and passing through  $(5, 7)$ .

23. Parallel to  $2x + y - 1 = 0$ , and passing through the intersection of  $3x + 2y - 59 = 0$  and  $5x - 7y + 6 = 0$ .

24. Parallel to the line joining  $(-2, 7)$  and  $(-4, -5)$ , and passing through  $(5, 3)$ .

25. Parallel to  $y = mx + b$ , and at a distance  $d$  from the origin.

26. Perpendicular to  $Ax + By + C = 0$ , and cutting an intercept  $b$  on the axis of  $y$ .

27. Perpendicular to  $\frac{x}{a} + \frac{y}{b} = 1$ , and passing through  $(a, b)$ .

28. Making the angle  $45^\circ$  with  $\frac{x}{a} + \frac{y}{b} = 1$ , and passing through  $(a, 0)$ .

29. Show that the triangle whose vertices are the points  $(2, 1)$ ,  $(3, -2)$ ,  $(-4, -1)$  is a right triangle.

30. The vertices of a triangle are  $(-1, -1)$ ,  $(-3, 5)$ ,  $(7, 11)$ . Find the equations of its altitudes. Prove that the altitudes meet in one point.

31. Find the equation of the perpendicular erected at the middle point of the line joining  $(5, 2)$  to the intersection of  $x + 2y - 11 = 0$  and  $9x - 2y - 59 = 0$ .

32. Find the equations of the perpendiculars erected at the middle points of the sides of the triangle whose vertices are  $(5, -7)$ ,  $(1, 11)$ ,  $(-4, 13)$ . Prove that these perpendiculars meet in one point.

33. The equations of the sides of a triangle are

$$x + y + 1 = 0, \quad 3x + 5y + 11 = 0, \quad x + 2y + 4 = 0.$$

Find (i.) the equations of the perpendiculars erected at the middle points of the sides; (ii.) the co-ordinates of their common point of intersection; (iii.) the distance of this point from the vertices of the triangle.

34. Show that the straight line passing through  $(a, b)$  and  $(c, d)$  is perpendicular to the straight line passing through  $(b, -a)$  and  $(d, -c)$ .

35. What is the equation of a straight line passing through  $(x_1, y_1)$ , and making an angle  $\phi$  with the line  $Ax + By + C = 0$ ?

#### DISTANCES.

48. Find the distance from the point  $(-4, 1)$  to the line  $3x - 4y + 1 = 0$ .

Ans. 3.

The required distance is the length of the perpendicular let fall from the given point to the given line. The first method that occurs for solving the problem is to form the equation of this perpendicular, find its intersection with the given line, and then compute the distance from this intersection to the given point.

Let this method be followed in solving the above problem and the first five problems of Ex. 15.

49. To find the distance from the point  $(x_1, y_1)$  to the line

$$x \cos \alpha + y \sin \alpha = p.$$

Let the line  $x \cos \alpha + y \sin \alpha = p'$ , (1)

which is evidently parallel to the given line, pass through the given point  $(x_1, y_1)$ ; then we have

$$x_1 \cos \alpha + y_1 \sin \alpha = p'.$$

Therefore  $x_1 \cos \alpha + y_1 \sin \alpha - p = p' - p$ .

But  $p' - p$  equals numerically the required distance. Hence the distance from the point  $(x_1, y_1)$  to the line  $x \cos \alpha + y \sin \alpha = p$  is obtained by substituting  $x_1$  for  $x$  and  $y_1$  for  $y$  in the expression  $x \cos \alpha + y \sin \alpha - p$ .

COR. 1. The distance as given by the formula will evidently be *positive* or *negative* according as the point and origin are on opposite sides of the line, or on the same side.

COR. 2. If the equation of the line be

$$Ax + By = C,$$

and  $d$  denote the distance from  $(x_1, y_1)$  to the line; then, evidently,

$$d = \frac{Ax_1 + By_1 - C}{\sqrt{A^2 + B^2}}. \quad [12]$$

Hence, to find the distance from the point  $(x_1, y_1)$  to the line  $Ax + By = C$ , write  $x_1$  for  $x$ , and  $y_1$  for  $y$  in the expression  $Ax + By - C$ , and divide the result by  $\sqrt{A^2 + B^2}$ .

For example, let  $(-1, 3)$  be the point, and  $2x + 4 = 3y$  the equation of the line.

Putting this in the form of [9], we have

$$-2x + 3y = 4.$$

Whence 
$$d = \frac{-2(-1) + 3 \times 3 - 4}{\sqrt{(-2)^2 + 3^2}} = +\frac{7}{13} \sqrt{13}.$$

Hence the point and origin are on opposite sides of the line. If only the length of  $d$  is sought, its sign may be neglected.



**Ex. 15.**

1. Find the distance from (1, 13) to the line  $3x = y - 5$ .
2. Find the distance from (8, 4) to the line  $y = 2x - 16$ .
3. Find the distance from the origin to the line  $3x + 4y = 20$ .
4. Find the distance from (2, 3) to the line  $2x + y - 4 = 0$ .
5. Find the distance from (3, 3) to the line  $y = 4x - 9$ .
6. Prove that the distance from the point  $(x_1, y_1)$  to the line  $y = mx + b$  is

$$d = \pm \frac{y_1 - mx_1 - b}{\sqrt{1 + m^2}},$$

the upper or lower sign being used according as  $b$  is positive or negative. Express this result in the form of a rule for practice.

7. Find the distances from the line  $3x + 4y + 15 = 0$  to the following points: (3, 0), (3, -1), (3, -2), (3, -3), (3, -4), (3, -5), (3, -6), (3, -7), (0, 0), (-1, 0), (-2, 0), (-3, 0), (-4, 0), (-5, 0), (-6, 0).

8. Find the distances from (1, 3) to the following lines:

$$3x + 4y + 15 = 0. \qquad 3x + 4y - 5 = 0.$$

$$3x + 4y + 10 = 0. \qquad 3x + 4y - 10 = 0.$$

$$3x + 4y + 5 = 0. \qquad 3x + 4y - 15 = 0.$$

$$3x + 4y = 0. \qquad 3x + 4y - 20 = 0.$$

Find the following distances:

9. From the point (2, -5) to the line  $y - 3x = 7$ .
10. From the point (4, 5) to the line  $4y + 5x = 20$ .
11. From the point (2, 3) to the line  $x + y = 1$ .
12. From the point (0, 1) to the line  $3x - 3y = 1$ .
13. From the point (-1, 3) to the line  $3x + 4y + 2 = 0$ .

14. From the origin to the line  $3x + 2y - 6 = 0$ .

15. From the point  $(2, -7)$  to the line joining  $(-4, 1)$  and  $(3, 2)$ .

16. From the line  $y = 7x$  to the intersection of the lines  $y = 3x - 4$  and  $y = 5x + 2$ .

17. From the origin to the line  $a(x - a) + b(y - b) = 0$ .

18. From the points  $(a, b)$  and  $(-a, -b)$  to the line

$$\frac{x}{a} + \frac{y}{b} = 1.$$

19. From the point  $(a, b)$  to the line  $ax + by = 0$ .

20. From the point  $(h, k)$  to the line  $Ax + By + C = D$ .

Find the distance between the two parallels :

21.  $3x + 4y + 15 = 0$  and  $3x + 4y + 5 = 0$ .

22.  $3x + 4y + 15 = 0$  and  $3x + 4y - 5 = 0$ .

23.  $Ax + By = C$  and  $Ax + By = C'$ .

24.  $Ax + By = C$  and  $-Ax - By = C'$ .

25.  $y = 5x - 7$  and  $y = 5x + 3$ .

26.  $\frac{x}{a} + \frac{y}{b} = 2$  and  $\frac{x}{a} + \frac{y}{b} = \frac{1}{2}$ .

27. Show that the locus of a point which is equidistant from the lines  $3x + 4y - 12 = 0$  and  $4x + 3y - 24 = 0$  consists of two straight lines. Find their equations, and draw a figure representing the four lines.

28. Show that the locus of a point which so moves that the sum of its distances from two given straight lines is constant is a straight line.

## AREAS.

50. To find the area of a triangle, having given its vertices.

SOLUTION I. Let  $PQR$  (Fig. 23) be the given triangle, and let the vertices of  $PQR$  be  $(x_1, y_1)$ ,  $(x_2, y_2)$ ,  $(x_3, y_3)$ , respectively. Drop the perpendiculars  $PM$ ,  $QN$ ,  $RL$ ; then

$$\text{area } PQR = PQNM + RLNQ - PMLR.$$

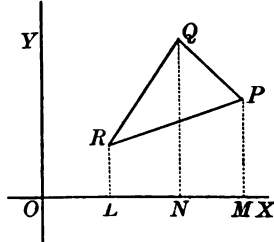


Fig. 23.

By Geometry,

$$\begin{aligned} PQNM &= \frac{1}{2} NM (MP + NQ) \\ &= \frac{1}{2} (x_1 - x_2) (y_1 + y_2). \end{aligned}$$

Similarly,

$$\begin{aligned} RLNQ &= \frac{1}{2} (x_2 - x_3) (y_2 + y_3), \\ PMLR &= \frac{1}{2} (x_1 - x_3) (y_1 + y_3). \end{aligned}$$

Substituting these values, we have

$$\begin{aligned} \text{area } PQR &= \frac{1}{2} [(x_1 - x_2) (y_2 + y_1) \\ &\quad + (x_2 - x_3) (y_3 + y_2) - (x_1 - x_3) (y_3 + y_1)] \\ &= \frac{1}{2} [-x_1 y_1 + x_1 y_2 - x_2 y_2 + x_2 y_3 - x_3 y_3 + x_3 y_1]. \\ \therefore \text{area} &= \frac{1}{2} [x_1 (y_2 - y_3) + x_2 (y_3 - y_1) + x_3 (y_1 - y_2)]. \quad [13] \end{aligned}$$

SOLUTION II. Since the area of a triangle is equal to one-half the product of its base and its altitude, this problem may be solved as follows:

- (i.) Find the length of either side as base.
- (ii.) Find the equation of the base.
- (iii.) Find the distance of the base from the opposite vertex.
- (iv.) Multiply this distance by one-half the base.

## Ex. 16.

Find the area of the triangle whose vertices are the points:

1.  $(0, 0)$ ,  $(1, 2)$ ,  $(2, 1)$ .
2.  $(3, 4)$ ,  $(-3, -4)$ ,  $(0, 4)$ .
3.  $(2, 3)$ ,  $(4, -5)$ ,  $(-3, -6)$ .
4.  $(8, 3)$ ,  $(-2, 3)$ ,  $(4, -5)$ .
5.  $(a, 0)$ ,  $(-a, 0)$ ,  $(0, b)$ .

6. Compare the formula for the area of a triangle with the result obtained by solving No. 68, p. 42. What, then, is the geometric meaning of that result?

Find the area of the figure having for vertices the points:

7.  $(3, 5), (7, 11), (9, 1)$ .
8.  $(3, -2), (5, 4), (-7, 3)$ .
9.  $(-1, 2), (4, 4), (6, -3)$ .
10.  $(0, 0), (x_1, y_1), (x_2, y_2)$ .
11.  $(2, -5), (2, 8), (-2, -5)$ .
12.  $(10, 5), (-2, 5), (-5, -3), (7, -3)$ .
13.  $(0, 0), (5, 0), (9, 11), (0, 3)$ .
14.  $(a, 1), (0, b), (c, 1)$ .
15.  $(a, b), (b, a), (c, c)$ .
16.  $(a, b), (b, a), (c, -c)$ .

17. Find the angles and the area of the triangle whose vertices are  $(3, 0), (0, 3\sqrt{3}), (6, 3\sqrt{3})$ .

What is the area contained by the lines

18.  $x=0, y=0, 5x+4y=20$ ?
19.  $x+y=1, x-y=0, y=0$ ?
20.  $x+2y=5, 2x+y=7, y=x+1$ ?
21.  $x+y=0, x=y, y=3a$ ?
22.  $y=3x, y=7x, y=c$ ?
23.  $x=0, y=0, x-4=0, y+6=0$ ?
24.  $3x+y+4=0, 3x-5y+34=0, 3x-2y+1=0$ ?
25.  $x-5y+13=0, 5x+7y+1=0, 3x+y-9=0$ ?
26.  $x-y=0, x+y=0, x-y=a, x+y=b$ ?

Find the area contained by the lines :

27.  $x = 0$ ,  $y = 0$ ,  $y = mx + b$ .

28.  $x = 0$ ,  $y = 0$ ,  $\frac{x}{a} + \frac{y}{b} = 1$ .

29.  $x = 0$ ,  $y = 0$ ,  $Ax + By + C = 0$ .

30.  $y = 3x - 9$ ,  $y = 3x + 5$ ,  $2y = x - 6$ ,  $2y = x + 14$ .

31. What is the area of the triangle formed by drawing straight lines from the point  $(2, 11)$  to the points in the line  $y = 5x - 6$  for which  $x = 4$ ,  $x = 7$ ?

**Ex. 17. (Review.)**

1. Deduce equation [7], p. 39, from equation [6].

2. The equation  $y = mx + b$  is not so general as the equation  $Ax + By + C = 0$ , because it cannot represent a line parallel to the axis of  $y$ . Explain more fully.

Determine for the following lines the values of  $a$ ,  $b$ ,  $\gamma$ ,  $p$ , and  $\alpha$  :

3.  $x = 2$ .

6.  $x + \sqrt{3}y = 2$ .

4.  $x = y$ .

7.  $x - \sqrt{3}y = 2$ .

5.  $y + 1 = \sqrt{3}(x + 2)$ .

8.  $\sqrt{3}x - y = 2$ .

9. Find the equations of the diagonals of the figure formed by the lines  $3x - y + 9 = 0$ ,  $3x = y - 1$ ,  $5x + 3y = 18$ ,  $5x + 3y = 2$ . What kind of quadrilateral is it? Why?

10. Find the distance between the parallels  $9x = y + 1$  and  $9x = y - 7$ .

11. The vertices of a quadrilateral are  $(3, 12)$ ,  $(7, 9)$ ,  $(2, -3)$ ,  $(-2, 0)$ . Find the equations of its sides and its area.

12. The vertices of a quadrilateral are  $(6, -4)$ ,  $(4, 4)$ ,  $(-4, 2)$ ,  $(-8, -6)$ . Prove that the lines joining the middle points of adjacent sides form a parallelogram. Find the area of this parallelogram.

Find the equation of a line passing through  $(3, 4)$ , and also

13. Perpendicular to the axis of  $x$ .
14. Making the angle  $45^\circ$  with the axis of  $x$ .
15. Parallel to the line  $5x + 6y + 8 = 0$ .
16. Intercepting on the axis of  $y$  the distance  $-10$ .
17. Passing through the point half way between  $(1, -4)$  and  $(-5, 4)$ .
18. Perpendicular to the line joining  $(3, 4)$  and  $(-1, 0)$ .

Find the equations of the following lines:

19. A line parallel to the line joining  $(x_1, y_1)$  and  $(x_2, y_2)$ , and passing through  $(x_3, y_3)$ .
20. The lines passing through  $(8, 3)$ ,  $(4, 3)$ ,  $(-5, -2)$ .
21. A line passing through the intersection of the lines  $2x + 5y + 8 = 0$  and  $3x - 4y - 7 = 0$ , and  $\perp$  to the latter line.
22. A line  $\perp$  to the line  $4x - y = 0$ , and passing through that point of the given line whose abscissa is 2.
23. A line  $\parallel$  to the line  $3x + 4y = 0$ , and passing through the intersection of the lines  $x - 2y - a = 0$  and  $x + 3y - 2a = 0$ .
24. A line through  $(4, 3)$ , such that the given point bisects the portion contained between the axes.
25. A line through  $(x_1, y_1)$ , such that the given point bisects the portion contained between the axes.
26. A line through  $(4, 3)$ , and forming with the axes in the second quadrant a triangle whose area is 8.
27. A line through  $(4, 3)$ , and forming with the axes in the fourth quadrant a triangle whose area is 8.
28. A line through  $(-4, 3)$ , such that the portion between the axes is divided by the given point in the ratio  $5 : 3$ .

29. A line dividing the distance between  $(-3, 7)$  and  $(5, -4)$  in the ratio  $4:7$ , and  $\perp$  to the line joining these points.

30. The two lines through  $(3, 5)$  making the angle  $45^\circ$  with the line  $2x - 3y - 7 = 0$ .

31. The two lines through  $(7, -5)$  which make the angle  $45^\circ$  with the line  $6x - 2y + 3 = 0$ .

32. The line making the angle  $45^\circ$  with the line joining  $(7, -1)$  and  $(-3, 5)$ , and intercepting the distance 5 on the axis of  $x$ .

33. The two lines which pass through the origin and trisect the portion of the line  $x + y = 1$  included between the axes.

34. The two lines  $\parallel$  to the line  $4x + 5y + 11 = 0$ , at the distance 3 from it.

35. The bisectors of the angles contained between the lines  $y = 2x + 4$ ,  $-y = 3x + 6$ .

HINT. Every point in the bisector of an angle is equidistant from the sides of the angle.

36. The bisectors of the angles contained between the lines  $2x - 5y = 0$ ,  $4x + 3y = 12$ .

37. The two lines which pass through  $(3, 12)$ , and whose distance from  $(7, 2)$  is equal to  $\sqrt{58}$ .

38. The two lines which pass through  $(-2, 5)$ , and are each equidistant from  $(3, -7)$  and  $(-4, 1)$ .

Find the angle contained between the lines:

39.  $y + 3 = 2x$  and  $y + 3x = 2$ .

40.  $y = 5x - 7$  and  $5y + x - 3 = 0$ .

Find the distance :

41. From the intersection of the lines  $3x + 2y + 4 = 0$ ,  $2x + 5y + 8 = 0$  to the line  $y = 5x + 6$ .

42. From the point  $(h, k)$  to the line  $\frac{x}{a} + \frac{y}{b} = 1$ .

43. From the origin to the line  $hx + ky = c^2$ .

44. From the point  $(a, 0)$  to the line  $y = mx + \frac{a}{m}$ .

Find the area included between the following lines :

45.  $x = y$ ,  $x + y = 0$ ,  $x = c$ .

46.  $x + y = k$ ,  $2x = y + k$ ,  $2y = x + k$ .

47.  $\frac{x}{a} + \frac{y}{b} = 1$ ,  $y = 2x + b$ ,  $x = 2y + a$ .

48.  $y = 4x + 7$  and the lines which join the origin to those points of the given line whose ordinates are  $-1$  and  $19$ .

49. The lines joining the middle points of the sides of the triangle formed by the lines  $x - 5y + 11 = 0$ ,  $11x + 6y - 1 = 0$ ,  $x + y + 4 = 0$ .

50. Find the area of the quadrilateral whose vertices are  $(0, 0)$ ,  $(0, 5)$ ,  $(11, 9)$ ,  $(7, 0)$ .

51. What point in the line  $5x - 4y - 28 = 0$  is equidistant from the points  $(1, 5)$  and  $(7, -3)$ ?

52. Prove that the diagonals of a square are  $\perp$  to each other.

53. Prove that the line joining the middle point of two sides of a triangle is parallel to the third side.

54. What is the geometric meaning of the equation  $xy = 0$ ?



55. Show that the three points  $(3a, 0)$ ,  $(0, 3b)$ ,  $(a, 2b)$  are in a straight line.

56. Show that the three lines  $5x + 3y - 7 = 0$ ,  $3x - 4y - 10 = 0$ , and  $x + 2y = 0$  meet in a point.

57. What must be the value of  $a$  in order that the three lines  $3x + y - 2 = 0$ ,  $2x - y - 3 = 0$ , and  $ax + 2y - 3 = 0$  may meet in a point?

What straight lines are represented by the equations:

58.  $x^2 + (a - b)x - ab = 0$ ?

59.  $xy + bx + ay + ab = 0$ ?

60.  $x^2y = xy^2$ ?

61.  $14x^2 - 5xy - y^2 = 0$ ?

In the following exercises prove that the locus of the point is a straight line, and obtain its equation.

62. The locus of the vertex of a triangle having the base and the area constant.

63. The locus of a point equidistant from the points  $(x_1, y_1)$  and  $(x_2, y_2)$ .

64. The locus of a point at the distance  $d$  from the line  $Ax + By + C = 0$ .

65. The locus of a point so moving that the sum of its distances from the axes shall be constant and equal to  $k$ .

66. The locus of a point so moving that the sum of its distances from the lines  $Ax + By + C = 0$ ,  $A'x + B'y + C' = 0$  shall be constant and equal to  $k$ .

67. The locus of the vertex of a triangle, having given the base and the difference of the squares of the other sides.

## SUPPLEMENTARY PROPOSITIONS.

## LINES PASSING THROUGH ONE POINT.

51. If  $S=0$ ,  $S'=0$  represent the equations of any two loci with the terms all transposed to the left-hand side, and  $k$  denotes an arbitrary constant, then the locus represented by the equation  $S+kS'=0$  passes through every point common to the two given loci.

For it is plain that any co-ordinates which satisfy the equation  $S=0$ , and also satisfy the equation  $S'=0$ , must likewise satisfy the equation  $S+kS'=0$ .

For what values of  $k$  will the equation  $S+kS'=0$  represent the lines  $S=0$  and  $S'=0$ , respectively?

52. Find the equation of the line joining the point (3, 4) to the intersection of the lines

$$3x - 2y + 17 = 0 \text{ and } x + 4y - 27 = 0.$$

The method of solving this question which first occurs is to find the intersection of the given lines and then apply equation [4], p. 37.

Another method, almost equally obvious, is to employ equation [5], which gives at once

$$y - 4 = m(x - 3),$$

and then determine  $m$  by substituting for  $x$  and  $y$  the co-ordinates of the intersection of the given lines.

The following method, founded on the principle stated in § 51, is, however, sometimes preferable, on account of its generality and because it saves the labor of solving the given equations. According to this principle, the required equation may be immediately written in the form

$$3x - 2y + 17 + k(x + 4y - 27) = 0.$$

And since the line passes through (3, 4), we must have

$$9 - 8 + 17 + k(3 + 16 - 27) = 0,$$

whence  $k = \frac{9}{4}.$

Therefore  $12x - 8y + 68 + 9x + 36y - 243 = 0,$

or  $3x + 4y - 25 = 0.$

This is the equation of the required line.

**53.** *If the equations of three straight lines are*

$$Ax + By + C = 0,$$

$$A'x + B'y + C' = 0,$$

$$A''x + B''y + C'' = 0,$$

*and we can find three constants,  $l, m, n$ , so that the relation  $l(Ax + By + C) + m(A'x + B'y + C') + n(A''x + B''y + C'') = 0$  is identically true, that is, true for all values of  $x$  and  $y$ , then the three lines meet in a point.*

For if the co-ordinates of any point satisfy any two of the equations, then the above relation shows that they will also satisfy the third equation.

**54.** *To find the equation of the bisector of the angle between the two lines*  $x \cos \alpha + y \sin \alpha = p,$

*and*  $x \cos \alpha' + y \sin \alpha' = p'.$

There are evidently two bisectors: one bisecting the angle in which the origin lies; the other bisecting the supplementary angle.

Now every point in either bisector is equally distant from the sides of the angle. Let  $(x, y)$  be any point in the bisector of the angle which includes the origin; then, by § 49, the expressions for its distances from the two sides will be equal numerically, and have like signs; hence we have

$$x \cos \alpha + y \sin \alpha - p = x \cos \alpha' + y \sin \alpha' - p'. \quad (1)$$

Since  $(x, y)$  is any point in this bisector, (1) is its equation.

The expressions for the distances of any point in the other bisector from the sides of the angle will be numerically equal, but unlike in sign; hence the equation of this bisector is

$$x \cos \alpha + y \sin \alpha - p = -(x \cos \alpha' + y \sin \alpha' - p'). \quad (2)$$

To distinguish equations (1) and (2) we note that in the first the constant terms in the two members have like signs; while in the second the constant terms have unlike signs.

COR. 1. If the equations of the lines are in the form

$$Ax + By = C, \quad A'x + B'y = C',$$

the equations of the bisectors are evidently

$$\frac{Ax + By - C}{\sqrt{A^2 + B^2}} = \pm \frac{A'x + B'y - C'}{\sqrt{A'^2 + B'^2}}. \quad [14]$$

Equation [14] represents the bisector of the angle in which the origin lies, or of its supplementary angle, *according as we take the upper or lower sign.*

For example, let the equations of the lines be

$$2x = 4y + 9, \text{ and } 5y = 3x - 7.$$

Putting these equations in the form of [9], we have

$$2x - 4y = 9, \text{ and } 3x - 5y = 7.$$

Hence the equations of the bisectors of their included angles are

$$\frac{2x - 4y - 9}{\sqrt{20}} = \pm \frac{3x - 5y - 7}{\sqrt{34}},$$

in which the upper sign gives the equation of the bisector of the angle in which the origin lies.

COR. 2. If  $S = 0$  and  $S' = 0$  represent two simple equations in the normal form, with the terms all transposed to the first members, then the equations of the bisectors of their included angles may be written

$$S = \pm S', \text{ or } S \mp S' = 0.$$

## Ex. 18.

Find the equation of a line passing through the intersection of the lines  $3x - 2y + 17 = 0$ ,  $x + 4y - 27 = 0$ , and

1. Passing also through the origin.
2. Parallel to the line  $x + 2y + 3 = 0$ .
3. Perpendicular to the line  $6x - 5y = 0$ .
4. Equally inclined to the two axes.
5. Find the equation of a line parallel to the line  $x = y$ , and passing through the intersection of the lines  $y = 2x + 1$  and  $y + 3x = 11$ .
6. Find the equation of a straight line joining  $(2, 3)$  to the intersection of the lines  
$$2x + 3y + 1 = 0 \text{ and } 3x - 4y = 5.$$
7. Find the equation of a straight line joining  $(0, 0)$  to the intersection of the lines  
$$5x - 2y + 3 = 0 \text{ and } 13x + y = 1.$$
8. Find the equation of a straight line joining  $(1, 11)$  to the intersection of the lines  
$$2x + 5y - 8 = 0 \text{ and } 3x - 4y = 8.$$

Find the equation of the straight line passing through the intersection of the lines  $Ax + By + C = 0$  and  $A'x + B'y + C' = 0$ , and also

9. Passing through the origin.
10. Drawn parallel to the axis of  $x$ .
11. Passing through the point  $(x_1, y_1)$ .
12. Find the equation of a straight line passing through the intersection of  $5x - 4y + 3 = 0$  and  $7x + 11y - 1 = 0$ , and cutting on the axis of  $y$  an intercept equal to 6.

13. Find the equation of a straight line passing through the intersection of  $y=7x-4$  and  $y=-2x+5$ , and forming with the axis of  $x$  the angle  $60^\circ$ .

14. The distance of a straight line from the origin is 5; and it passes through the intersection of the lines  $3x-2y+11=0$  and  $6x+7y-55=0$ . What is its equation?

15. What is the equation of the straight line passing through the intersection of  $bx+ay=ab$  and  $y=mx$ , and perpendicular to the former line?

Prove that the following lines are concurrent (or pass through one point):

16.  $y=2x+1$ ,  $y=x+3$ ,  $y=-5x+15$ .

17.  $4x-2y-3=0$ ,  $3x-y+\frac{1}{2}=0$ ,  $5x-2y-1=0$ .

18.  $2x-y=5$ ,  $3x-y=6$ ,  $4x-y=7$ .

19. What is the value of  $m$  if the lines

$$\frac{x}{a} + \frac{y}{b} = 1, \quad \frac{x}{b} + \frac{y}{a} = 1, \quad y = mx$$

meet in one point?

20. When do the straight lines  $y=mx+b$ ,  $y=m'x+b'$ ,  $y=m''x+b''$  pass through one point?

21. Prove that the three altitudes of a triangle meet in one point.

22. Prove that the perpendiculars erected at the middle points of the sides of a triangle meet in one point.

23. Prove that the three medians of a triangle meet in one point. Show also that this point is one of the two points of trisection for each median.

24. Prove that the bisectors of the three angles of a triangle meet in one point.

25. The vertices of a triangle are  $(2, 1)$ ,  $(3, -2)$ ,  $(-4, -1)$ . Find the lengths of its altitudes. Is the origin within or without the triangle?

26. The equations of the sides of a triangle are

$$3x + y + 4 = 0, \quad 3x - 5y + 34 = 0, \quad 3x - 2y + 1 = 0.$$

Find the lengths of its altitudes.

What are the equations of the lines bisecting the angles between the lines

27.  $3x - 4y + 7 = 0$  and  $4x - 3y + 17 = 0$ ?

28.  $3x + 4y - 9 = 0$  and  $12x + 5y - 3 = 0$ ?

29.  $y = 2x - 4$  and  $2y = x + 10$ ?

30.  $x + y = 2$  and  $x - y = 0$ ?

31.  $y = mx + b$  and  $y = m'x + b'$ ?

32. Prove that the bisectors of the two supplementary angles formed by two intersecting lines are perpendicular to each other.

#### EQUATIONS REPRESENTING STRAIGHT LINES.

55. *A homogeneous equation of the  $n$ th degree represents  $n$  straight lines through the origin.*

Let the equation be

$$Ax^n + Bx^{n-1}y + Cx^{n-2}y^2 + \dots + Ky^n = 0.$$

Dividing by  $Ay^n$ , we have

$$\left(\frac{x}{y}\right)^n + \frac{B}{A}\left(\frac{x}{y}\right)^{n-1} + \frac{C}{A}\left(\frac{x}{y}\right)^{n-2} + \dots + \frac{K}{A} = 0.$$

If  $r_1, r_2, r_3, \dots, r_n$  denote the roots of this equation, then the equation, resolved into its factors, becomes

$$\left(\frac{x}{y} - r_1\right)\left(\frac{x}{y} - r_2\right)\left(\frac{x}{y} - r_3\right) \dots \left(\frac{x}{y} - r_n\right) = 0,$$

and therefore is satisfied when any one of these factors is zero, and in no other cases.

Therefore the locus of the equation consists of the  $n$  straight lines

$$x - r_1 y = 0, \quad x - r_2 y = 0, \quad \dots, \quad x - r_n y = 0.$$

56. To find the angle between the two straight lines represented by the equation  $Ax^2 + Cxy + By^2 = 0$ .

Solving the equation as a quadratic in  $x$ , we obtain

$$2Ax + (C \pm \sqrt{C^2 - 4AB})y = 0.$$

Hence the slopes of the two lines are

$$m = \frac{2A}{-C - \sqrt{C^2 - 4AB}}, \quad m' = \frac{2A}{-C + \sqrt{C^2 - 4AB}}.$$

Therefore

$$m - m' = \frac{\sqrt{C^2 - 4AB}}{B}, \quad mm' = \frac{A}{B};$$

and (equation [10], p. 45),

$$\tan \phi = \frac{m - m'}{1 + mm'} = \frac{\sqrt{C^2 - 4AB}}{A + B}.$$

57. To find the condition that the general equation of the second degree may represent two straight lines.

We may write the most general form of the equation of the second degree as follows :

$$Ax^2 + By^2 + Cxy + Dx + Ey + F = 0. \quad (1)$$

In order that this equation may represent two straight lines, its first member must be the product of two linear factors in  $x$  and  $y$ ; that is, the equation can be written in the form

$$(lx + my + n)(px + qy + r) = 0. \quad (2)$$

Equating coefficients in (1) and (2), we obtain

$$\begin{aligned} lp &= A, & mq &= B, & nr &= F. \\ lq + mp &= C, & lr + np &= D, & mr + nq &= E. \end{aligned}$$



The product of  $C$ ,  $D$ , and  $E$  is

$$\begin{aligned} CDE &= 2lmnpqr + lp(n^2q^2 + m^2r^2) + mq(l^2r^2 + n^2p^2) \\ &\quad + nr(m^2p^2 + l^2q^2) \\ &= 2ABF + A(E^2 - 2BF) + B(D^2 - 2AF) \\ &\quad + F(C^2 - 2AB). \end{aligned}$$

Hence the required condition is

$$F(C^2 - 4AB) + AE^2 + BD^2 - CDE = 0. \quad (3)$$

### Ex. 19.

1. Describe the position of the two straight lines represented by the equation  $Ax^2 + Cxy + By^2 + Dx + Ey + F = 0$ , where (i.)  $A = C = D = 0$ , (ii.)  $B = C = E = 0$ .

2. When will the equation  $axy + bx + cy + d = 0$  represent two straight lines?

3. Find the conditions that the straight lines represented by the equation  $Ax^2 + Cxy + By^2 = 0$  may be real; imaginary; coincident; perpendicular to each other.

4. Show that the two straight lines  $x^2 - 2xy \sec \theta + y^2 = 0$  make the angle  $\theta$  with each other.

Show that the following equations represent straight lines, and find their separate equations:

5.  $x^2 - 2xy - 3y^2 + 2x - 2y + 1 = 0$ .

6.  $x^2 - 4xy + 5y^2 - 6y + 9 = 0$ .

7.  $x^2 - 4xy + 3y^2 + 6y - 9 = 0$ .

8. Show that the equation  $x^2 + xy - 6y^2 + 7x + 31y - 18 = 0$  represents two straight lines, and find the angle between them.

Determine the values of  $K$  for which the following equations will represent in each case a pair of straight lines. Are the lines real or imaginary?

9.  $12x^2 - 10xy + 2y^2 + 11x - 5y + K = 0$ .

$$10. 12x^2 + Kxy + 2y^2 + 11x - 5y + 2 = 0.$$

$$11. 12x^2 + 36xy + Ky^2 + 6x + 6y + 3 = 0.$$

12. For what value of  $K$  does the equation  $Kxy + 5x + 3y + 2 = 0$  represent two straight lines?

#### PROBLEMS ON LOCI INVOLVING THREE VARIABLES.

**58.** *A trapezoid is formed by drawing a line parallel to the base of a given triangle. Find the locus of the intersection of its diagonals.*

If  $ABC$  be the given triangle, and we choose for axes the base  $AB$  and the altitude  $CO$ , the vertices  $A, B, C$  may be represented in general by  $(a, 0), (b, 0), (0, c)$ , respectively. The equations of  $AC$  and  $BC$  are

$$\frac{x}{a} + \frac{y}{c} = 1 \quad \text{and} \quad \frac{x}{b} + \frac{y}{c} = 1.$$

Let  $y = m$  be the equation of the line parallel to the base, and let it cut  $AC$  in  $D$ ,  $BC$  in  $E$ ; then the co-ordinates of  $D$  and  $E$ , respectively, are

$$\left( \frac{ac - am}{c}, m \right) \quad \text{and} \quad \left( \frac{bc - bm}{c}, m \right).$$

Hence the equation of the diagonal  $BD$  is

$$\frac{y}{x - b} = \frac{cm}{ac - am - bc}, \quad (1)$$

and the equation of the diagonal  $AE$  is

$$\frac{y}{x - a} = \frac{cm}{bc - bm - ac}. \quad (2)$$

If  $P$  be the intersection of the diagonals, then the co-ordinates  $x$  and  $y$  of the point  $P$  must satisfy both (1) and (2); by solving these equations, therefore, we obtain *for any particular value of  $m$*  the co-ordinates of the point  $P$ . But what we want is the algebraic relation which is satisfied by the

co-ordinates of  $P$ , *whatever the value of  $m$  may be*. To find this, we have only to eliminate  $m$  from equations (1) and (2). By doing this we obtain

$$2cx + (a+b)y = (a+b)c,$$

or

$$\frac{x}{\frac{1}{2}(a+b)} + \frac{y}{c} = 1.$$

We see from the form of this equation that the required locus is the line which joins  $C$  to the middle point of  $AB$ .

**REMARK.** The above solution should be studied till it is understood. In problems on loci it is often necessary to obtain relations which involve not only the  $x$  and  $y$  of a point of the locus which we are seeking, but also *some third variable* (as  $m$  in the above example).

In such cases we must obtain *two* equations which involve  $x$  and  $y$  and this third variable, and then eliminate the third variable; the resulting equation will be the equation of the locus required.

#### Ex. 20.

1. Through a fixed point  $O$  any straight line is drawn, meeting two given parallel straight lines in  $P$  and  $Q$ ; through  $P$  and  $Q$  straight lines are drawn in fixed directions, meeting in  $R$ . Prove that the locus of  $R$  is a straight line, and find its equation.

2. The hypotenuse of a right triangle slides between the axes of  $x$  and  $y$ , its ends always touching the axes. Find the locus of the vertex of the right angle.

3. Given two fixed points,  $A$  and  $B$ , one on each of the axes; if  $U$  and  $V$  are two variable points, one on each axis, so taken that  $OU + OV = OA + OB$ , find the locus of the intersection of  $AV$  and  $BU$ .

4. Find the locus of the middle points of the rectangles which may be inscribed in a given triangle.

5. If  $PP'$ ,  $QQ'$  are any two parallels to the sides of a given rectangle, find the locus of the intersection of  $P'Q$  and  $PQ'$ .

## CHAPTER III.

### THE CIRCLE.

#### EQUATIONS OF THE CIRCLE.

59. The **Circle** is the locus of a point which moves so that its distance from a fixed point is constant. The fixed point is the **centre**, and the constant distance the **radius**, of the circle.

NOTE. The word "circle," as here defined, means the same thing as "circumference" in Elementary Geometry. This is the usual meaning of "circle" in the higher branches of Mathematics.

60. To find the equation of a circle, having given its centre  $(a, b)$  and its radius  $r$ .

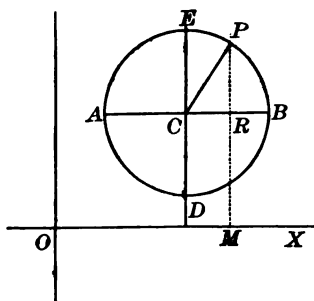


Fig. 24.

Let  $C$  (Fig. 24) be the centre, and  $P$  any point  $(x, y)$  of the circumference. Then it is only necessary to express by an equation the fact that the distance from  $P$  to  $C$  is constant, and equal to  $r$ : the required equation evidently is (§ 6)

$$(x - a)^2 + (y - b)^2 = r^2. \quad [15]$$

If we draw  $CR \parallel$  to  $OX$ , to meet the ordinate of  $P$ , then we see from the figure that the legs of the rt.  $\triangle CPR$  are  $CR = x - a$ ,  $PR = y - b$ .

If the origin be taken at the centre, then  $a = b = 0$ , and the equation of the circle is

$$x^2 + y^2 = r^2. \quad [16]$$

This is the simplest form of the equation of a circle, and the one most commonly used.

If the origin be taken on the circumference at the point  $A$ , and the diameter  $AB$  be taken as the axis of  $x$ , then the centre will be the point  $(r, 0)$ . Writing  $r$  in place of  $a$ , and 0 in place of  $b$  in [15], and reducing, we obtain

$$x^2 + y^2 = 2rx. \quad [17]$$

Why is this equation without any constant term?

**61.** *The locus of any equation of the second degree in  $x$  and  $y$  in which the term in  $xy$  is wanting and the coefficients of  $x^2$  and  $y^2$  are equal is a circle.*

Any such equation can evidently be reduced to the form

$$x^2 + y^2 + 2Dx + 2Ey + F = 0. \quad (1)$$

Therefore

$$\begin{aligned} & (x^2 + 2Dx + D^2) + (y^2 + 2Ey + E^2) = D^2 + E^2 - F, \\ \text{or} \quad & (x + D)^2 + (y + E)^2 = (D^2 + E^2 - F). \end{aligned} \quad (2)$$

Now from [15] it follows that the locus of (2) is a circle whose centre is  $(-D, -E)$ , and whose radius is

$$\sqrt{D^2 + E^2 - F}.$$

**COR.** If  $D^2 + E^2 > F$ , the radius is real and the circle is readily constructed. If  $D^2 + E^2 = F$ , the radius is zero, and the locus is the single point  $(-D, -E)$ . If  $D^2 + E^2 < F$ , the radius is imaginary, and the equation represents no real locus.

62. Any point  $(h, k)$  is without, on, or within the circle  $x^2 + y^2 = r^2$ , according as  $h^2 + k^2 >, =$ , or  $< r^2$ .

For a point is without, on, or within a circle, according as its distance from the centre  $>, =$ , or  $<$  the radius.

### Ex. 21.

Find the equation of the circle, taking as origin

1. The point  $B$  (Fig. 24).
2. The point  $D$  (Fig. 24).
3. The point  $E$  (Fig. 24).

Write the equations of the following circles :

4. Centre  $(5, -3)$ , radius 10.
5. Centre  $(0, -2)$ , radius 11.
6. Centre  $(5, 0)$ , radius 5.
7. Centre  $(-5, 0)$ , radius 5.
8. Centre  $(2, 3)$ , diameter 10.
9. Centre  $(h, k)$ , radius  $\sqrt{h^2 + k^2}$ .
10. Determine the centre and radius of the circle

$$x^2 + y^2 - 10x + 12y + 25 = 0.$$

Here  $(x - 5)^2 + (y + 6)^2 = 36$ .  $\therefore a = 5, b = -6, r = 6$ .

Determine the centres and radii of the following circles :

- |                                       |                                      |
|---------------------------------------|--------------------------------------|
| 11. $x^2 + y^2 - 2x - 4y = 0$ .       | 17. $6x^2 - 2y(7 - 3y) = 0$ .        |
| 12. $3x^2 + 3y^2 - 5x - 7y + 1 = 0$ . | 18. $x^2 + y^2 = 9k^2$ .             |
| 13. $x^2 + y^2 - 8x = 0$ .            | 19. $(x + y)^2 + (x - y)^2 = 8k^2$ . |
| 14. $x^2 + y^2 + 8x = 0$ .            | 20. $x^2 + y^2 = a^2 + b^2$ .        |
| 15. $x^2 + y^2 - 8y = 0$ .            | 21. $x^2 + y^2 = k(x + k)$ .         |
| 16. $x^2 + y^2 + 8y = 0$ .            | 22. $x^2 + y^2 = hx + ky$ .          |

23. When are the circles  $x^2 + y^2 + Dx + Ey + C = 0$  and  $x^2 + y^2 + D'x + E'y + C' = 0$  concentric?

24. What is the geometric meaning of the equation  $(x-a)^2 + (y-b)^2 = 0$ ?

25. Find the intercepts of the circles

$$(i.) x^2 + y^2 - 8x - 8y + 7 = 0,$$

$$(ii.) x^2 + y^2 - 8x - 8y + 16 = 0,$$

$$(iii.) x^2 + y^2 - 8x - 8y + 20 = 0.$$

Putting  $y = 0$  in each case, we have in case (i.)  $x^2 - 8x + 7 = 0$ , whence  $x = 1$  and  $7$ ; in case (ii.)  $x^2 - 8x + 16 = 0$ , whence  $x = 4$ ; in case (iii.)  $x^2 - 8x + 20 = 0$ , whence  $x = \pm \sqrt{-4}$ .

Putting  $x = 0$  in each case, we obtain for  $y$  values identical with the above values of  $x$ .

The geometric meaning of these results is as follows:

Circle (i.) cuts the axis of  $x$  in the points  $(1, 0)$ ,  $(7, 0)$ , and the axis of  $y$  in the points  $(0, 1)$ ,  $(0, 7)$ .

Circle (ii.) touches the axis of  $x$  at  $(4, 0)$ , and the axis of  $y$  at  $(0, 4)$ .

Circle (iii.) does not meet the axes at all.

This is the meaning of the imaginary values of  $x$  and  $y$  in case (iii.).

If, however, we wished to make the language of Geometry conform more exactly to that of Algebra, then in this case we should say that the circle meets the axes in imaginary points. This form of statement, however, must be understood as simply another way of saying that the circle does not meet the axes.

Find the centres, radii, and intercepts on the axes of the following circles:

$$26. x^2 + y^2 - 5x - 7y + 6 = 0.$$

$$27. x^2 + y^2 - 12x - 4y + 15 = 0.$$

$$28. x^2 + y^2 - 4x - 8y = 0.$$

$$29. x^2 + y^2 - 6x + 4y + 4 = 0.$$

$$30. x^2 + y^2 + 22x - 18y + 57 = 0.$$

31. Under what conditions will the circle  $x^2 + y^2 + Dx + Ey + C = 0$  (i.) touch the axis of  $x$ ? (ii.) touch the axis of  $y$ ? (iii.) not meet the axes at all?

32. Show that the circle  $x^2 + y^2 + 10x - 10y + 25 = 0$  touches the axes and lies entirely in the second quadrant. Write the equation so that it shall represent the same circle touching the axes and lying in the third quadrant.

33. In what points does the straight line  $3x + y = 25$  cut the circle  $x^2 + y^2 = 65$ ?

34. Find the points common to the loci  $x^2 + y^2 = 4$  and  $y = 2x - 4$ .

35. The equation of a chord of the circle  $x^2 + y^2 = 25$  is  $y = 2x + 11$ . Find its length.

36. The equation of a chord is  $\frac{x}{a} + \frac{y}{b} = 1$ ; that of the circle is  $x^2 + y^2 = r^2$ . Find the length of the chord.

37. Find the equation of a line passing through the centre of  $x^2 + y^2 - 6x - 8y = -21$  and perpendicular to  $x + 2y = 4$ .

38. Find the equation of that chord of the circle  $x^2 + y^2 = 130$  which passes through the point for which the abscissa is 9 and the ordinate negative, and which is  $\parallel$  to the straight line  $4x - 5y - 7 = 0$ .

39. What is the equation of the chord of the circle  $x^2 + y^2 = 277$  which passes through  $(3, -5)$  and is bisected at this point?

40. Find the locus of the centre of a circle passing through the points  $(x_1, y_1)$  and  $(x_2, y_2)$ .

41. What is the locus of the centres of all the circles which pass through the points  $(5, 3)$  and  $(-7, -6)$ ?



Find the equation of a circle :

42. Passing through the points (4, 0), (0, 4), (6, 4).
43. Passing through the points (0, 0), (8, 0), (0, -6).
44. Passing through the points (-6, -1), (0, 0), (0, -1).
45. Passing through the points (0, 0), (-8a, 0), (0, 6a).
46. Passing through the points (2, -3), (3, -4), (-2, -1).
47. Passing through the points (1, 2), (1, 3), (2, 5).
48. Passing through (10, 4) and (17, -3), and radius = 13.
49. Passing through (3, 6), and touching the axes.
50. Touching each axis at the distance 4 from the origin.
51. Touching each axis at the distance  $a$  from the origin.
52. Passing through the origin, and cutting the lengths  $a$ ,  $b$  from the axes.
53. Passing through (5, 6), and having its centre at the intersection of the lines  $y = 7x - 3$ ,  $4y - 3x = 13$ .
54. Passing through (10, 9) and  $(5, 2 - 3\sqrt{6})$ , and having its centre in the line  $3x - 2y - 17 = 0$ .
55. Passing through the origin, and cutting equal lengths  $a$  from the lines  $x = y$ ,  $x + y = 0$ .
56. Circumscribing the triangle whose sides are the lines  $y = 0$ ,  $y = mx + b$ ,  $\frac{x}{a} + \frac{y}{b} = 1$ .
57. Having for diameter the line joining (0, 0) and  $(x_1, y_1)$ .
58. Having for diameter the line joining  $(x_1, y_1)$  and  $(x_2, y_2)$ .
59. Having for diameter the line joining the points where  $y = mx$  meets  $x^2 + y^2 = 2rx$ .
60. Having for diameter the common chord of the circles  $x^2 + y^2 = r^2$  and  $(x - a)^2 + y^2 = r^2$ .

## TANGENTS AND NORMALS.

63. Let  $QPQ'$  (Fig. 25) represent any curve. If the secant  $QPR$  be turned about the point  $P$  until the point  $Q$  approaches indefinitely near to  $P$ , then the limiting position,  $TT'$ , of the secant is called the **Tangent** to the curve at  $P$ .

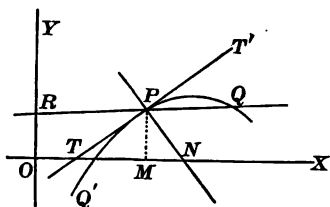


Fig. 25.

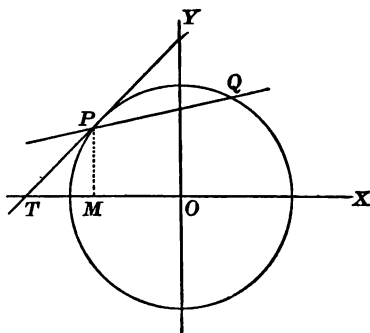


Fig. 26.

The tangent  $TT'$  is said to *touch* the curve at  $P$ , and the point  $P$  is called the **Point of Contact**.

The straight line  $PN$  drawn from  $P$ , perpendicular to the tangent  $TT'$ , is called the **Normal** to the curve at  $P$ .

Let the curve be referred to the axes  $OX$ ,  $OY$ , and let  $M$  be the foot of the ordinate of the point  $P$ . Let also the tangent and the normal at  $P$  meet the axis of  $x$  in the points  $T$ ,  $N$ , respectively. Then  $TM$  is called the **Subtangent** for the point  $P$ , and  $MN$  is called the **Subnormal**.

64. To find the equation of a tangent to the circle  $x^2 + y^2 = r^2$ , at the point of contact  $(x_1, y_1)$ .

Let  $P$  (Fig. 26) be the point  $(x_1, y_1)$ , and  $Q$  any other point  $(x_2, y_2)$  of the circle. Then the equation of the secant  $PQ$  is

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}. \quad (1)$$

Now, since  $(x_1, y_1)$  and  $(x_2, y_2)$  are on this circle, we have

$$x_1^2 + y_1^2 = r^2,$$

$$x_2^2 + y_2^2 = r^2,$$

Subtracting,  $(x_2^2 - x_1^2) + (y_2^2 - y_1^2) = 0.$

Factoring,  $(x_2 - x_1)(x_2 + x_1) + (y_2 - y_1)(y_2 + y_1) = 0.$

Whence, by transposition and division, we have

$$\frac{y_2 - y_1}{x_2 - x_1} = -\frac{x_2 + x_1}{y_2 + y_1}.$$

By substituting in (1), the equation of the secant becomes

$$\frac{y - y_1}{x - x_1} = -\frac{x_2 + x_1}{y_2 + y_1}.$$

Now let  $Q$  coincide with  $P$ , or  $x_2 = x_1, y_2 = y_1$ ; the secant becomes a tangent at  $P$ , and the equation becomes

$$\frac{y - y_1}{x - x_1} = -\frac{x_1}{y_1},$$

or  $x_1x + y_1y = x_1^2 + y_1^2.$

And, since  $x_1^2 + y_1^2 = r^2$ , we obtain

$$x_1x + y_1y = r^2, \quad [18]$$

which is the equation required.

NOTE. If we had put  $x_2 = x_1, y_2 = y_1$ , in (1) before we introduced the condition that  $(x_1, y_1)$  and  $(x_2, y_2)$  were on the circle, the slope of the tangent would have assumed the indeterminate form  $\frac{0}{0}$ .

The above method of obtaining the equation of the tangent to a circle is applicable to any curve whatever. It is sometimes called the *secant* method. Equation [18] is easily remembered from its symmetry, and because it may be formed from  $x^2 + y^2 = r^2$  by merely changing  $x^2$  to  $x_1x$ , and  $y^2$  to  $y_1y$ .

**65.** *To find the equation of the normal through  $(x_1, y_1)$ .*

The slope of the tangent is  $-\frac{x_1}{y_1}$ .

Therefore that of the normal will be  $\frac{y_1}{x_1}$  (§ 45, Cor. 2).

Hence the equation of the normal is (§ 38)

$$y - y_1 = \frac{y_1}{x_1} (x - x_1),$$

which reduces to the form

$$y_1 x - x_1 y = 0. \quad [19]$$

Therefore the normal passes through the centre.

**66.** *To find the equations of the tangent and normal to the circle  $(x - a)^2 + (y - b)^2 = r^2$  at the point of contact  $(x_1, y_1)$ .*

We proceed as in § 64, only now the equations of condition which place  $(x_1, y_1)$  and  $(x_2, y_2)$  on the circle are

$$\begin{aligned} (x_1 - a)^2 + (y_1 - b)^2 &= r^2, \\ (x_2 - a)^2 + (y_2 - b)^2 &= r^2. \end{aligned}$$

After subtracting and factoring, we have

$$(x_2 - x_1)(x_2 + x_1 - 2a) + (y_2 - y_1)(y_2 + y_1 - 2b) = 0,$$

whence 
$$\frac{y_2 - y_1}{x_2 - x_1} = -\frac{x_2 + x_1 - 2a}{y_2 + y_1 - 2b}.$$

Hence the equation of a secant through  $(x_1, y_1)$  and  $(x_2, y_2)$  is

$$\frac{y - y_1}{x - x_1} = -\frac{x_2 + x_1 - 2a}{y_2 + y_1 - 2b}.$$

Making  $x_2 = x_1$ , and  $y_2 = y_1$ , and reducing, we obtain

$$(x_1 - a)(x - a) + (y_1 - b)(y - b) = r^2. \quad [20]$$

Equation [20] may be immediately formed from [18] by affixing  $-a$  to the  $x$  factors and  $-b$  to the  $y$  factors, on the left-hand side.

By proceeding as in § 65, we obtain for the equation of the normal

$$(y_1 - b)(x - x_1) - (x_1 - a)(y - y_1) = 0. \quad [21]$$

67. To find the condition that the straight line  $y = mx + c$  shall touch the circle  $x^2 + y^2 = r^2$ .

I. If the line touch the circle, it is evident that the perpendicular from the origin to the line must be equal the radius  $r$  of the circle. The length of this perpendicular is  $\frac{c}{\sqrt{1+m^2}}$  (§ 41). Therefore the required condition is expressed by the equation

$$c^2 = r^2(1 + m^2).$$

II. By eliminating  $y$  from the equations

$$y = mx + c, \quad x^2 + y^2 = r^2,$$

we obtain the quadratic in  $x$ ,

$$(1 + m^2)x^2 + 2mcx = r^2 - c^2,$$

the two roots of which are

$$x = -\frac{mc}{1 + m^2} \pm \frac{\sqrt{r^2(1 + m^2) - c^2}}{1 + m^2}.$$

If these roots are real and unequal, the line will *cut* the circle; if they are equal, it will *touch* the circle; if they are imaginary, it will *not meet the circle at all*.

The roots will be equal if  $\sqrt{r^2(1 + m^2) - c^2} = 0$ ; that is, if  $c^2 = r^2(1 + m^2)$ , a result agreeing with that previously obtained.

If in the equation  $y = mx + c$  we substitute for  $c$  the value  $r\sqrt{1 + m^2}$ , we obtain the equation to the tangent of a circle in the useful form

$$y = mx \pm r\sqrt{1 + m^2}. \quad [22]$$



*Find the portion of the tangent  
between the axes.*

This equation, if we regard  $m$  as an arbitrary constant, represents all possible tangents to the circle  $x^2 + y^2 = r^2$ .

NOTE 1. Method II. is applicable to any curve, and agrees with the definition of a tangent given in § 63.

NOTE 2. In problems on tangents the learner should consider whether the co-ordinates of the point of contact are involved. If they are, he should use equation [18]; if they are not, then in general it is better to use equation [22].

Ex. 22.

1. Explain the meaning of the double sign in equation [22].
2. Deduce the equations of the tangent and normal to the circle  $x^2 + y^2 = r^2$ , assuming that the normal passes through the centre.
3. Find the equations of the tangent and the normal to  $x^2 + y^2 = 52$  at the point (4, 6). Find subtangent and subnormal, and the portions of the tangent and normal between  $P$  (Fig. 25) and the axis of  $x$ , which are sometimes called the tangent and normal. 
4. A straight line touches the circle  $x^2 + y^2 = r^2$  in the point  $(x_1, y_1)$ . Find the lengths of the subtangent, the subnormal, and the portion of the line ~~contained~~ between the axes. 
5. What is the equation of a tangent to the circle  $x^2 + y^2 = 250$  at the point whose abscissa is 9 and ordinate negative?
6. Find the equations of tangents to  $x^2 + y^2 = 10$  at the points whose common abscissa = 1.
7. Tangents are drawn through the points of the circle  $x^2 + y^2 = 25$  which have abscissas numerically equal to 3. Prove that these tangents enclose a rhombus, and find its area.
8. The subtangent for a certain point of a circle is  $5\frac{1}{2}$ ; the subnormal is 3. What is the equation of the circle?

Find the equation of a straight line

9. Touching  $x^2 + y^2 = 232$  at the point whose abscissa = 14.
10. Touching  $(x - 2)^2 + (y - 3)^2 = 10$  at the point (5, 4).
11. Touching  $x^2 + y^2 - 3x - 4y = 0$  at the origin.
12. Touching  $x^2 + y^2 - 14x - 4y - 5 = 0$  at the point whose abscissa is equal to 10.

What is the equation of a straight line touching the circle  $x^2 + y^2 = r^2$ , and also

13. Passing through the point of contact  $(r, 0)$ ?
14. Parallel to the line  $Ax + By + C = 0$ ?
15. Perpendicular to the line  $Ax + By + C = 0$ ?
16. Making the angle  $45^\circ$  with the axis of  $x$ ?
17. Passing through the exterior point  $(h, 0)$ ?
18. Forming with the axes a triangle of area  $r^2$ ?
19. Find the equations of the tangents drawn from the point (10, 5) to the circle  $x^2 + y^2 = 100$ .
20. Find the equations of tangents to the circle  $x^2 + y^2 + 10x - 6y - 2 = 0$  and  $\parallel$  to the line  $y = 2x - 7$ .
21. Find the lengths of subtangent and subnormal in the circle  $x^2 + y^2 - 14x - 4y = 5$  for the point (10, 9).
22. What is the equation of the circle (centre at origin) which is touched by the straight line  $x \cos \alpha + y \sin \alpha = p$ ? What are the co-ordinates of the point of contact?
23. When will the line  $Ax + By - C = 0$  touch the circle  $x^2 + y^2 = r^2$ ? the circle  $(x - a)^2 + (y - b)^2 = r^2$ ?
24. Find the equation of a straight line touching  $x^2 + y^2 = ax + by$  and passing through the origin.

Prove that the following circles and straight line touch, and find the points of contact in each case :

25.  $x^2 + y^2 + ax + by = 0$  and  $ax + by + a^2 + b^2 = 0$ .

26.  $x^2 + y^2 - 2ax - 2by + b^2 = 0$  and  $x = 2a$ .

27.  $x^2 + y^2 = ax + by$  and  $ax - by + b^2 = 0$ .

28. What is the equation of the circle (centre at origin) which touches the line  $y = 3x - 5$ ?

29. What must be the value of  $m$  in order that the line  $y = mx + 10$  may touch the circle  $x^2 + y^2 = 100$ ? Show that we get the same answer for the line  $y = mx - 10$ , and explain the reason.

30. Determine the value of  $c$  in order that the line  $3x - 4y + c = 0$  may touch the circle  $x^2 + y^2 - 8x + 12y - 44 = 0$ . Explain the double answer.

31. What is the equation of the circle having for centre the point  $(5, 3)$  and touching the line  $3x + 2y - 10 = 0$ ?

32. What is the equation of a circle whose radius = 10, and which touches the line  $4x + 3y - 70 = 0$  in the point  $(10, 10)$ ?

33. A circle touching the line  $4x + 3y + 3 = 0$  in the point  $(-3, 3)$  passes through the point  $(5, 9)$ . What is its equation?

34. Under what condition will the line  $\frac{x}{a} + \frac{y}{b} = 1$  touch the circle  $x^2 + y^2 = r^2$ ?

35. What is the equation of the circle inscribed in the triangle whose sides are

$$x = 0, \quad y = 0, \quad \frac{x}{a} + \frac{y}{b} = 1?$$



36. Two circles touch each other when the distance between their centres is equal to the sum or the difference of their radii. Prove that the circles

$$x^2 + y^2 = (r + a)^2, \quad (x - a)^2 + y^2 = r^2$$

touch each other, and find the equation of the common tangent.

37. Two circles touch each other when the length of their common chord = 0. Find the length of the common chord of

$$(x - a)^2 + (y - b)^2 = r^2, \quad (x - b)^2 + (y - a)^2 = r^2,$$

and hence prove that the two circles touch each other when  $(a - b)^2 = 2r^2$ .

**Ex. 23. (Review.)**

Find the radii and centres of the following circles:

1.  $3x^2 - 6x + 3y^2 + 9y - 12 = 0$ .
2.  $7x^2 + 3y^2 - 4y - (1 - 2x)^2 = 0$ .
3.  $y(y - 5) = x(3 - x)$ .
4.  $\sqrt{1 + a^2}(x^2 + y^2) = 2b(x + ay)$ .

Find the equation of a circle:

5. Centre (0, 0), radius = 9.
6. Centre (7, 0), radius = 3.
7. Centre (-2, 5), radius = 10.
8. Centre (3a, 4a), radius = 5a.
9. Centre (b + c, b - c), radius = c.
10. Passing through (a, 0), (0, b), (2a, 2b).
11. Passing through (0, 0), (0, 12), (5, 0).
12. Passing through (10, 9), (4, -5), (0, 5).
13. Touching each axis at the distance -7 from the origin.

14. Touching both axes, and radius =  $r$ .

15. Centre  $(a, a)$ , and cutting chord =  $b$  from each axis.

16. Having the centre  $(0, 0)$ , and touching  $y = 2x + 3$ .

17. Having the centre  $(1, -3)$ , and touching  $2x - y - 4 = 0$ .

18. With its centre in the line  $5x - 7y - 8 = 0$ , and touching the lines  $2x - y = 0$ ,  $x - 2y - 6 = 0$ .

19. Passing through the origin and the points common to the circles

$$x^2 + y^2 - 6x - 10y - 15 = 0,$$

$$x^2 + y^2 + 2x + 4y - 20 = 0.$$

20. Having its centre in the line  $5x - 3y - 7 = 0$ , and passing through the points common to the same circles as in No. 19.

21. Touching the axis of  $x$ , and passing through the points common to the circles

$$x^2 + y^2 + 4x - 14y - 68 = 0,$$

$$x^2 + y^2 - 6x - 22y + 30 = 0.$$

22. Find the centre and the radius of the circle which passes through  $(9, 6)$ ,  $(10, 5)$ ,  $(3, -2)$ .

23. What is the distance from the centre of the circle passing through  $(2, 0)$ ,  $(8, 0)$ ,  $(5, 9)$  to the straight line joining  $(0, -11)$  and  $(-16, 1)$ ?

24. What is the distance from the centre of the circle  $x^2 + y^2 - 4x + 8y = 0$  to the line  $4x - 3y + 30 = 0$ ?

25. What portion of the line  $y = 5x + 2$  is contained within the circle  $x^2 + y^2 - 13x - 4y - 9 = 0$ ?

26. Through that point of the circle  $x^2 + y^2 = 25$  for which the abscissa = 4 and the ordinate is negative, a straight line parallel to  $y = 3x - 5$  is drawn. Find the length of the intercepted chord.

27. Through the point  $(x_1, y_1)$ , within the circle  $x^2 + y^2 = r^2$ , a chord is drawn so as to be bisected at this point. What is its equation?

28. What relation must exist among the coefficients of the equation  $A(x^2 + y^2) + Dx + Ey + C = 0$ .

(i.) in order that the circle may touch the axis of  $x$ ?

(ii.) in order that the circle may touch the axis of  $y$ ?

(iii.) in order that the circle may touch both axes?

29. Under what condition will the straight line  $y = mx + c$  touch the circle  $x^2 + y^2 = 2rx$ ?

30. What must be the value of  $k$  in order that the line  $3x + 4y = k$  may touch the circle  $y^2 = 10x - x^2$ ?

31. Find the equation of the circle which passes through the origin and cuts equal lengths  $a$  from the lines  $x = y$ ,  $x + y = 0$ .

32. Find the equations of the four circles whose common radius  $= a\sqrt{2}$ , and which cut chords from each axis equal to  $2a$ .

33. Find the equation of the circle whose diameter is the common chord of the circles  $x^2 + y^2 = r^2$ ,  $(x - a)^2 + y^2 = r^2$ .

Find the equation of the straight line

34. Passing through  $(0, 0)$  and the centre of the circle

$$x^2 + y^2 = a(x + y).$$

35. Passing through the centres of the circles

$$x^2 + y^2 = 25 \text{ and } x^2 + y^2 + 6x - 8y = 0.$$

36. Passing through  $(0, 0)$  and touching the circle

$$x^2 + y^2 - 6x - 12y + 41 = 0.$$

37. Parallel to  $x + \sqrt{3}(y - 12) = 0$  and touching  $x^2 + y^2 = 100$ .

38. Passing through the points common to the circles

$$x^2 + y^2 - 2x - 4y - 20 = 0,$$

$$x^2 + y^2 - 14x - 16y + 100 = 0.$$

39. Prove that the common chord of the circles in No. 38 is perpendicular to the straight line joining their centres.

40. Find the area of the triangle formed by radii of the circle  $x^2 + y^2 = 169$  drawn to the points whose abscissas are  $-12$  and  $+7$  and ordinates positive, and the chord passing through the same two points.

41. Prove that an angle inscribed in a semicircle is a right angle.

42. Prove that the radius of a circle drawn perpendicular to a chord bisects the chord.

43. Find the inclination to the axis of  $x$  of the line joining the centres of the circles  $x^2 + 2x + y^2 = 0$ ,  $x^2 + 2y + y^2 = 0$ .

44. Determine the point from which tangents drawn to the circles

$$x^2 + y^2 - 2x - 6y + 6 = 0,$$

$$x^2 + y^2 - 22y - 20x + 52 = 0,$$

will each be equal to  $4\sqrt{6}$ .

45. Find the equations of the circles which touch the straight lines  $6x + 7y + 9 = 0$  and  $7x + 6y + 3 = 0$ , and the latter line in the point  $(3, -4)$ .

Obtain and discuss the equations of the following loci :

46. Locus of the centres of a circle having the radius  $r$  and passing through the point  $(x_1, y_1)$ .

47. Locus of the centre of a circle having the radius  $r'$  and touching the circle  $(x - a)^2 + (y - b)^2 = r^2$ .

48. Locus of all points from which tangents drawn to the circle  $(x - a)^2 + (y - b)^2 = r^2$  have a given length  $l$ .

49. Locus of the middle point of a chord drawn through a fixed point  $A$  of a given circle.

50. Locus of the point  $M$  which divides the chord  $AC$ , drawn through the fixed point  $A$  of a given circle, in a given ratio  $AM:MC = m:n$ .

51. Locus of a point whose distances from two fixed points,  $A, B$ , are in a constant ratio  $m:n$ .

52. Locus of a point, the sum of the squares of whose distances from two fixed points,  $A$  and  $B$ , is constant, and equal to  $k^2$ .

53. Locus of a point, the difference of the squares of whose distances from two fixed points,  $A, B$ , is constant and equal to  $k^2$ .

54. Locus of the middle point of a line of constant length  $d$  which moves so that its ends always touch two fixed perpendicular lines.

55. Locus of the vertex of a triangle whose base is fixed and of constant length, and the angle at the vertex is also constant.

56. One side,  $AB$ , of a triangle is constant in length and fixed in position; another side,  $AC$ , is constant in length but revolves about the point  $A$ . Find the locus of the middle point of the third side,  $BC$ .

57. Find the locus of the intersections of tangents at the extremities of a chord whose length is constant.

58. Given the equation of a circle  $x^2 + y^2 = r^2$ . If its radii are produced, each by a length equal to the abscissa of the point where it meets the circle, find the locus of the extremities of the radii produced.

## SUPPLEMENTARY PROPOSITIONS.

68. A **Diameter** of a curve is the locus of the middle points of a system of parallel chords. The chords which any diameter bisects are called the **Chords** of that diameter.

69. To find the equation of a diameter of the circle  $x^2 + y^2 = r^2$ .

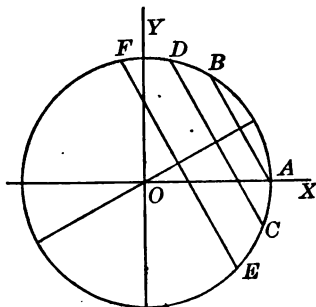


Fig. 27.

Let the equation of any one of the parallel chords (Fig. 27) be  $y = mx + c$ , and let it meet the circle in the points  $(x_1, y_1)$  and  $(x_2, y_2)$ .

$$\text{Then (§§ 37 and 64)} \quad m = -\frac{x_1 + x_2}{y_1 + y_2}. \quad (1)$$

Let  $(x, y)$  be the middle point of the chord; then  $2x = x_1 + x_2$ ,  $2y = y_1 + y_2$  (§ 8), and by substitution we have

$$m = -\frac{x}{y},$$

or

$$y = -\frac{1}{m}x, \quad (2)$$

a relation which evidently holds true for the middle points of all the chords. Therefore (2) is the equation of a diameter.

COR. From (2) we see that a diameter of a circle is a straight line passing through the centre and perpendicular to its chords.

70. *Two distinct, two coincident, or no tangents can be drawn to a circle through any point  $(h, k)$ , according as this point is without, on, or within the circle.*

Let the tangent

$$y = mx + r\sqrt{1+m^2}$$

pass through the point  $(h, k)$ ; then

$$k = mh + r\sqrt{1+m^2}.$$

Transposing and squaring, we have

$$(h^2 - r^2)m^2 - 2hkm = r^2 - k^2.$$

$$\therefore m = \frac{hk \pm r\sqrt{h^2 + k^2 - r^2}}{h^2 - r^2}. \quad (1)$$

The values of  $m$  given in (1) are the slopes of the tangents through  $(h, k)$ . Now, these values are real and unequal, equal, or imaginary, according as  $h^2 + k^2 >, =, \text{ or } < r^2$ ; that is, according as  $(h, k)$  is without, on, or within the circle. Hence two distinct, two coincident, or no tangents can be drawn through  $(h, k)$ , according as  $(h, k)$  is without, on, or within the circle.

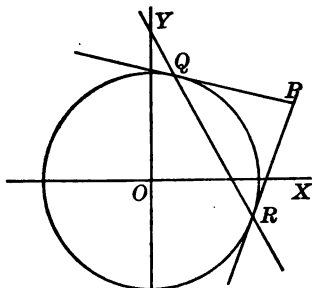


Fig. 28.

71. *To find the equation of the chord joining the points of contact of two tangents from any external point  $(h, k)$ .*

Let  $(x_1, y_1), (x_2, y_2)$  be the points of contact  $Q$  and  $R$ ; then the equations of the tangents  $PQ$  and  $PR$  are (§ 64)

$$x_1x + y_1y = r^2,$$

$$x_2x + y_2y = r^2.$$

Since both tangents pass through  $P(h, k)$ , both these equations are satisfied by the co-ordinates  $h, k$ ; therefore

$$hx_1 + ky_1 = r^2, \quad (1)$$

$$hx_2 + ky_2 = r^2. \quad (2)$$

From equations (1) and (2) we see that the co-ordinates of both the points  $(x_1, y_1), (x_2, y_2)$  satisfy the equation

$$hx + ky = r^2. \quad (3)$$

Hence the locus of (3), which is a right line, passes through both points of contact; and therefore (3) is the equation of the chord  $QR$ . The chord  $QR$  is called the **Chord of Contact**.

**72.** *Suppose a chord of a circle to turn round any fixed point  $(h, k)$ ; to find the locus of the intersection of the two tangents drawn at its extremities.*

Let  $P$  (Fig. 29 or 30) be the fixed point  $(h, k)$ ,  $QPR$  one position of the revolving chord, and let the tangents at  $Q$  and  $R$  intersect in  $P_1(x_1, y_1)$ ; it is required to find the locus of  $P_1$  as the chord turns about  $P$ . Since  $QR$  is the chord of contact of tangents drawn from the point  $P_1(x_1, y_1)$ , its equation is (§ 71)

$$x_1x + y_1y = r^2. \quad (1)$$

Since (1) passes through  $(h, k)$ , we have

$$hx_1 + ky_1 = r^2. \quad (2)$$

But  $(x_1, y_1)$  is *any* point in the required locus, and by (2) its co-ordinates satisfy the equation

$$hx + ky = r^2; \quad (3)$$

hence (3) is the equation of the required locus.

Since (3) is of the first degree, the locus is a straight line.





**COR. 1.** If the point  $(h, k)$  be on the circle, (3) is evidently the equation of the tangent at  $(h, k)$ ; hence,

*The polar of any point on the circle is identical with the tangent at that point.*

**COR. 2.** If  $(h, k)$  be an external point, by § 71, (3) is the equation of the chord of contact of tangents from  $(h, k)$  to the circle; hence,

*The polar of any external point is the same line as the chord of contact of tangents drawn from that point.*

Thus, in Fig. 30,  $HM$  is the polar of  $P$ , or the chord of contact of tangents drawn from  $P$ .

**73.** The polar and pole of a circle may be defined as follows: If a chord of a circle be turned round a fixed point  $(h, k)$ , the locus of the intersection of the two tangents at its extremities is the polar of the pole  $(h, k)$  with regard to that curve.

**74.** *If the polar of a point  $P$  pass through  $P'$ , then the polar of  $P'$  will pass through  $P$ .*

Let  $P$  be the point  $(h, k)$ ,  $P'$  the point  $(h', k')$ , and let the equation of the circle be  $x^2 + y^2 = r^2$ .

Then the equation of the polars of  $P$  and  $P'$  are

$$hx + ky = r^2, \quad (1)$$

$$h'x + k'y = r^2. \quad (2)$$

If  $P'$  be on the polar of  $P$ , its co-ordinates must satisfy equation (1); therefore

$$hh' + kk' = r^2.$$

But this is also the condition that  $P$  shall be on the line represented by (2); that is, on the polar of  $P'$ . Therefore  $P$  is on the polar of  $P'$ .

This relation of poles and polars is illustrated in the Figs. 29 and 30.

75. To find a geometrical construction for the polar of a point with respect to a circle.

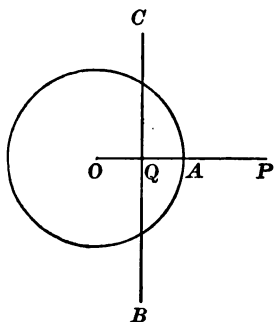


Fig. 31.

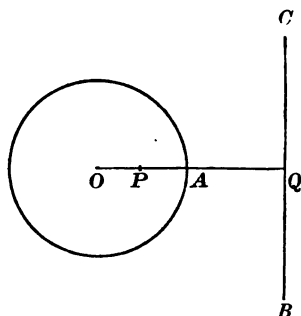


Fig. 32.

The equation of the line through any point  $P(h, k)$  and the centre of the circle, or the origin, is

$$kx - hy = 0. \quad (1)$$

Now the equation of the polar of  $P$  is

$$hx + ky = r^2. \quad (2)$$

But the loci of (1) and (2) are perpendicular (§ 45, Cor. 2).

Hence, if  $BC$  is the polar of  $P$ ,  $OP$  is perpendicular to  $BC$ , and

$$OQ = \frac{r^2}{\sqrt{h^2 + k^2}}. \quad \S 41$$

Also

$$OP = \sqrt{h^2 + k^2}.$$

Therefore

$$OP \times OQ = r^2.$$

Hence, to construct the polar of  $P$ :

Join  $OP$ , and let it cut the circle in  $A$ ; take  $Q$  in the line  $OP$ , so that

$$OP : OA = OA : OQ.$$

The line through  $Q$  perpendicular to  $OP$  is the polar of  $P$ .

To locate the pole of  $BC$ , draw  $OQ$  perpendicular to  $BC$ , and take  $P$  so that

$$OQ : OA = OA : OP.$$

76. To find the length of the tangent drawn from any point  $(h, k)$  to the circle  $(x - a)^2 + (y - b)^2 - r^2 = 0$ . (1)

Let  $P$  (Fig. 33) be the point  $(h, k)$ ,  $Q$  the point of contact,  $C$  the centre of the circle; then, since  $PQC$  is a right angle,

$$PQ^2 = PC^2 - QC^2.$$

Now  $PC^2 = (h - a)^2 + (k - b)^2$ ,  $QC^2 = r^2$ .

Therefore  $PQ^2 = (h - a)^2 + (k - b)^2 - r^2$ .

Hence  $PQ^2$  is found by simply substituting the co-ordinates of  $P$  for  $x$  and  $y$  in the expression  $(x - a)^2 + (y - b)^2 - r^2$ .

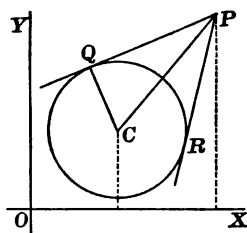


Fig. 33.

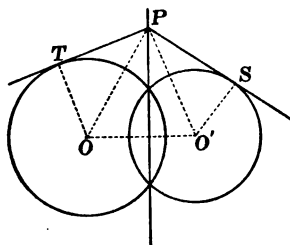


Fig. 34.

If for brevity we write  $S$  instead of  $(x - a)^2 + (y - b)^2 - r^2$ , then the equation  $S = 0$  will represent the general equation of the circle after division by the common coefficient of  $x^2$  and  $y^2$ , and we may state the above result as follows:

*If  $S = 0$  be the equation of a circle, and the co-ordinates of any point be substituted for  $x$  and  $y$  in  $S$ , the result will be equal to the square of the length of the tangent drawn from the point to the circle.*

77. To find the locus of the point from which tangents drawn to two given circles are equal.

Let the equations of the circles  $O$  and  $O'$  (Fig. 34), be

$$(x - a)^2 + (y - b)^2 - r^2 = 0, \quad (1)$$

$$(x - a')^2 + (y - b')^2 - r'^2 = 0. \quad (2)$$

Then, if the tangents drawn from  $P(x, y)$  to the circles (1) and (2) are equal, we have

$$(x-a)^2 + (y-b)^2 - r^2 = (x-a')^2 + (y-b')^2 - r'^2, \quad (3)$$

which is the equation of the required locus.

COR. 1. Performing the indicated operations in (3), and transposing, we have

$$2(a-a')x + 2(b-b')y = a^2 - a'^2 + b^2 - b'^2 - r^2 + r'^2, \quad (4)$$

which shows that the locus is a straight line.

This locus is called the **Radical Axis** of the circles (1) and (2).

Hence, if  $S_1 = 0$ ,  $S_2 = 0$  be the equations of two circles, then

$$S_1 = S_2, \text{ or } S_1 - S_2 = 0,$$

will be the equation of their radical axis.

COR. 2. When the circles  $S_1 = 0$  and  $S_2 = 0$  intersect, the locus of  $S_1 = S_2$  passes through their common points.

Hence, when two circles intersect or are tangent, their radical axis is their common chord or tangent.

COR. 3. The slope of (4) is the negative reciprocal of the slope of the line joining the centres of (1) and (2).

Hence, the radical axis of two circles is perpendicular to the line joining their centres.

78. Let  $S = 0$ ,  $S_1 = 0$ ,  $S_2 = 0$  be the equations of three circles, in each of which the coefficient of  $x^2$  is unity.

Then the equations of their radical axes, taken in pairs, are

$$S - S_1 = 0, \quad S_1 - S_2 = 0, \quad S - S_2 = 0.$$

The values of  $x$  and  $y$  that will satisfy any two of these equations will also satisfy the third. Therefore the third axis passes through the point common to the other two. Hence,

The three radical axes of three circles, taken in pairs, meet in a point. This point is called the **Radical Centre** of the three circles,

## Ex. 24.

1. What is the equation of the diameter of the circle  $x^2 + y^2 = 20$  which bisects chords parallel to the line  $6x + 7y + 8 = 0$ ?

2. What is the equation of the diameter of the circle which bisects all chords whose inclination to the axis of  $x$  is  $135^\circ$ ?

3. Prove that the tangents at the extremities of a diameter are parallel.

4. Write the equations of the chords of contact in the circle  $x^2 + y^2 = r^2$  for tangents drawn from the following points:  $(r, r)$ ,  $(2r, 3r)$ ,  $(a + b, a - b)$ .

5. From the point  $(13, 2)$  tangents are drawn to the circle  $x^2 + y^2 = 49$ ; what is the equation of the chord of contact?

6. What line is represented by the equation  $hx + ky = r^2$  when  $(h, k)$  is in the circle?

7. Write the equations of the polars of the following points with respect to the circle  $x^2 + y^2 = 4$ :

(i.)  $(2, 3)$ .      (ii.)  $(3, -1)$ .      (iii.)  $(1, -1)$ .

8. Find the poles of the following lines with respect to the circle  $x^2 + y^2 = 35$ :

(i.)  $4x + 6y = 7$ .      (ii.)  $3x - 2y = 5$ .      (iii.)  $ax + by = 1$ .

9. Find the pole of  $3x + 4y = 7$  with respect to the circle  $x^2 + y^2 = 14$ .

10. Find the pole of  $Ax + By + C = 0$  with respect to the circle  $x^2 + y^2 = r^2$ .

11. Find the co-ordinates of the points where the line  $x = 4$  cuts the circle  $x^2 + y^2 = 25$ ; also find the equations of the tangents at those points, and show that they intersect in the point  $(\frac{25}{4}, 0)$ .

12. If the polars of two points  $P, Q$  meet in  $R$ , then  $R$  is the pole of the line  $PQ$ .

13. If the polar of  $(h, k)$  with respect to the circle  $x^2 + y^2 = r^2$  touch the circle  $x^2 + y^2 = 2rx$ , then  $k^2 + 2rh = r^2$ .

14. If the polar of  $(h, k)$  with respect to the circle  $x^2 + y^2 = c^2$  touch the circle  $4(x^2 + y^2) = c^2$ , then the pole  $(h, k)$  will lie on the circle  $x^2 + y^2 = 4c^2$ .

15. Find the polar of the centre of the circle  $x^2 + y^2 = r^2$ . Trace the changes in the position of the polar as the pole is supposed to move from the centre to an infinite distance.

16. What is the square of the tangent drawn from the point  $(h, k)$  to the circle  $x^2 + y^2 = r^2$ ?

17. Find the length of the tangent drawn from  $(2, 5)$  to the circle  $x^2 + y^2 - 2x - 3y - 1 = 0$ .

Find the radical axis of the circles

18.  $(x + 5)^2 + (y + 6)^2 = 9$ ,  $(x - 7)^2 + (y - 11)^2 = 16$ .

19.  $x^2 + y^2 + 2x + 3y - 7 = 0$ ,  $x^2 + y^2 - 2x - y + 1 = 0$ .

20.  $x^2 + y^2 + bx + by - c = 0$ ,  $ax^2 + ay^2 + a^2x + b^2y = 0$ .

21. Find the radical axis and length of the common chord of the circles

$$x^2 + y^2 + ax + by + c = 0, \quad x^2 + y^2 + bx + ay + c = 0.$$

22. Find the radical centre of the three circles

$$x^2 + y^2 + 4x + 7 = 0,$$

$$2x^2 + 2y^2 + 3x + 5y + 9 = 0,$$

$$x^2 + y^2 + y = 0.$$

## CHAPTER IV.

### DIFFERENT SYSTEMS OF CO-ORDINATES.

#### RECTILINEAR SYSTEM.

79. When we define the position of a point, with reference to any fixed lines or points, we are said to use a **System of Co-ordinates**.

In the **Rectilinear System**, already described, we have thus far employed only rectangular axes, or co-ordinates, which are to be preferred for most purposes, on account of their greater simplicity. When the axes of reference intersect at oblique angles, the axes and co-ordinates are called **Oblique**.

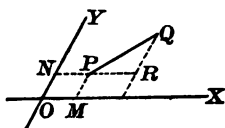


Fig. 35.

Let  $OX, OY$  (Fig. 35) be two axes making an acute angle,  $\angle XOY = \omega$ , with each other. If we draw  $PN \parallel$  to  $OX$ , and  $PM \parallel$  to  $OY$ , then the co-ordinates of  $P$  are

$$NP = OM = x, \quad MP = y.$$

Since oblique and rectangular co-ordinates differ only in the angle included between the axes, any of the previously deduced formulas which do not depend on any property of the right angle, are applicable when the axes are oblique. Thus, formulas [2], [3], [4], [7] hold for oblique axes as well



as for rectangular, and therefore are general formulas in the rectilinear system.

When the axes are oblique, instead of [1], we evidently have (Fig. 35)

$$PQ = \sqrt{PR^2 + RQ^2 - 2PR \times RQ \cos PRQ},$$

$$\therefore d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + 2(x_2 - x_1)(y_2 - y_1) \cos \omega},$$

which reduces to [1] when  $\omega = 90^\circ$ .

The Rectilinear System is sometimes called the **Cartesian System**, from Descartes, who first used it.

80. To find the equation of the straight line  $AC$ , referred to the oblique axes  $OX, OY$  (Fig. 36), having given the intercept  $OB = b$  and the angle  $XAC = \gamma$ .

Let  $P$  be any point  $(x, y)$  of the line. Draw  $BD \parallel$  to  $OX$ , meeting  $PM$  in  $D$ . Then, by Trigonometry,

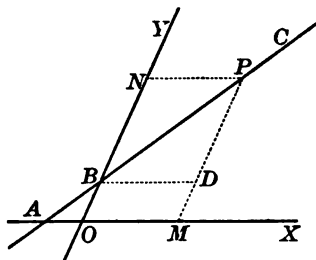


Fig. 36.

$$\frac{PD}{BD} = \frac{\sin \gamma}{\sin(\omega - \gamma)}, \text{ or } \frac{y - b}{x} = \frac{\sin \gamma}{\sin(\omega - \gamma)}$$

If now we put  $m = \frac{\sin \gamma}{\sin(\omega - \gamma)}$ , we obtain as the result an equation of the same form as [6], p. 38,

$$y = mx + b.$$

Here  $m$  = the ratio of the sines of the angles which the line  $AC$  makes with the axes; that is,  $m = \sin XAP + \sin PBY$ , which equals  $\tan XAP = \tan \gamma$  when  $\omega = 90^\circ$ .

**81.** Oblique co-ordinates are seldom used, because they generally lead to more complex formulas than rectangular ones. In certain cases, however, they may be employed to advantage. An example of this kind is furnished by problem No. 23, p. 65 :

*To prove that the medians of a triangle meet in one point.*

If  $a, b, c$  represent the three sides of the triangle, and we take as axes the sides  $a$  and  $b$ , then the equations of the sides and also of the medians may be written down with great ease, as follows :

The sides,  $y=0, \quad x=0, \quad \frac{x}{a} + \frac{y}{b} = 1.$

The medians,

$$\frac{2x}{a} + \frac{y}{b} - 1 = 0, \quad \frac{x}{a} + \frac{2y}{b} - 1 = 0, \quad \frac{x}{a} - \frac{y}{b} = 0.$$

On comparing the equations of the medians, we see that if we subtract the second equation from the first, we obtain the third; therefore the three medians must pass through the same point (§ 53).

#### POLAR SYSTEM OF CO-ORDINATES.

**82.** Next to the rectilinear, the system of co-ordinates most frequently used is the **Polar System**.

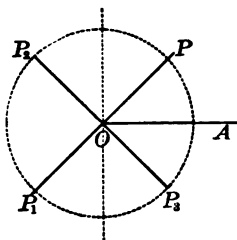


Fig. 37.

Let  $O$  (Fig. 37) be a fixed point,  $OA$  a fixed straight line,  $P$  any point. Join  $OP$ .

It is evident that we know the position of  $P$ , provided we know the distance  $OP$  and the angle which  $OP$  forms with  $OA$ .

Thus, if we denote the distance  $OP$  by  $\rho$ , and the angle  $AOP$  by  $\theta$ , the position of  $P$  is determined if  $\rho$  and  $\theta$  are known.

The fixed point  $O$  is called the **Pole**, and the fixed line  $OA$  the **Polar Axis**;  $\rho$  and  $\theta$  are called the **Polar Co-ordinates** of  $P$ ;  $\rho$ , its **Radius Vector**; and  $\theta$ , its **Direction** or **Vectorial Angle**.

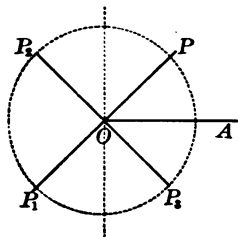


Fig. 38.

Every point in a plane is perfectly determined by a positive value of  $\rho$  between 0 and  $\infty$ , and a positive value of  $\theta$  between  $0^\circ$  and  $360^\circ$  (or 0 and  $2\pi$ , circular measure). But, in order to represent by a single equation all the points of a geometric locus, we often employ negative values of  $\rho$  and  $\theta$ , and adopt the following laws of signs:

(i.)  $\theta$  is positive when measured from right to left, and negative when estimated in the opposite direction.

(ii.)  $\rho$  is positive or negative according as it extends in the direction of the terminal side of  $\theta$  or in the opposite direction. Thus any given point may be determined in four different ways.

For example, suppose that the straight line  $POP_1$  bisects the first and third quadrants, and that in this line we take points  $P, P_1$ , at the same distance  $OP = a$  from  $O$ ; then

$P$  is the point  $(a, \frac{1}{4}\pi)$  or  $(-a, \frac{5}{4}\pi)$  or  $(-a, -\frac{3}{4}\pi)$  or  $(a, -\frac{7}{4}\pi)$ ;

$P_1$  is the point  $(a, \frac{3}{4}\pi)$  or  $(-a, \frac{1}{4}\pi)$  or  $(a, -\frac{5}{4}\pi)$  or  $(-a, -\frac{7}{4}\pi)$ .

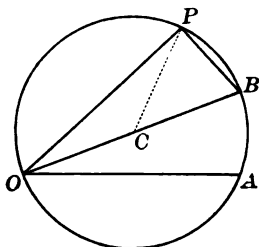


Fig. 39.

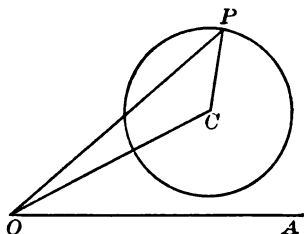


Fig. 40.

### 83. To find the polar equation of the circle.

(i.) Let the pole  $O$  be at the centre. Then, if  $r$  denote the radius, the polar equation is simply  $\rho = r$ .

(ii.) Let the pole  $O$  be on the circumference (Fig. 39), and let the diameter  $OB$  make an angle  $\alpha$  with the initial line  $OA$ . Let  $P$  be any point  $(\rho, \theta)$  of the circle. Join  $BP$ .

Then  $OP = OB \cos BOP$ ,

$$\text{or} \quad \rho = 2r \cos(\theta - \alpha). \quad [23]$$

If  $OB$  is taken as the initial line, the equation becomes

$$\rho = 2r \cos \theta. \quad [24]$$

(iii.) Let the pole  $O$  be anywhere, and the centre the point  $(\rho', \theta')$ . Then in the triangle  $OCP$  (Fig. 40),

$$OP^2 - 2OP \times OC \times \cos COP + OC^2 - CP^2 = 0,$$

$$\text{or} \quad \rho^2 - 2\rho\rho' \cos(\theta - \theta') + \rho'^2 - r^2 = 0, \quad [25]$$

the most general form of the polar equation of a circle.

## Ex. 25.

1. Find the distances from the point  $P$  in Fig. 38 to the two axes.

2. Prove that the equation of a straight line, referred to oblique axes in terms of its intercepts, is identical in form with [7], p. 39.

3. If the straight line  $P_2OP_3$  (Fig. 38) bisects the second and fourth quadrants, what are the polar co-ordinates of the points  $P_2$  and  $P_3$ ? Give more than one set of values in each case.

4. Construct the following points (on paper, take  $a = 1$  in.):

$$\begin{aligned} & \left(a, 0\right), \left(a, \frac{\pi}{2}\right), \left(a, -\frac{\pi}{2}\right), \left(-a, \frac{\pi}{2}\right), \left(-a, -\frac{\pi}{2}\right), \\ & \left(2a, \frac{\pi}{6}\right), \left(2a, \pi\right), \left(a \cos \frac{\pi}{3}, \frac{\pi}{3}\right), \left(a, \frac{3\pi}{2}\right), \left(3a, \frac{2\pi}{3}\right), \\ & \left(-3a, \frac{2\pi}{3}\right), \left(4a, \tan^{-1} \frac{4}{3}\right), \left(4a, \tan^{-1} \frac{3}{4}\right). \end{aligned}$$

NOTE. The expression  $\tan^{-1} \frac{4}{3}$  in higher Mathematics means "the angle whose tangent is  $\frac{4}{3}$ ."

5. If  $\rho_1, \rho_2$  denote the two values of  $\rho$  in equation [25], p. 103, prove that  $\rho_1\rho_2 = \rho^2 - r^2$ . What theorem of Elementary Geometry is expressed by this equation (i.) when the pole is outside the circle? (ii.) when the pole is inside the circle?

6. Through a fixed point  $P$  in a circle a chord  $PB$  is drawn, and then revolved about  $P$ ; find the locus of its middle point.

NOTE. In such problems as this there is a great advantage in using polar equations.

7. If  $p$  denote the distance from the pole to a straight line,  $a$  the angle between  $p$  and the polar axis, prove that the polar equation of the line is  $\rho \cos(\theta - a) = p$ .

## TRANSFORMATION OF CO-ORDINATES.

84. The equation of a curve is oftentimes greatly simplified by referring it to a new set of axes, or to a new system of co-ordinates. For example, compare equations [15] and [16], p. 71. Hence, it is sometimes useful to be able to deduce from the equation of a curve referred to one set of axes or to one system of co-ordinates, its equation when referred to another set of axes or to another system of co-ordinates. Either of these operations is known as the **Transformation of Co-ordinates**. It consists in expressing the old co-ordinates in terms of the new, and then replacing in the equation of the curve the old co-ordinates by their values in terms of the new; we thus obtain a constant relation between the new co-ordinates, which will represent the curve referred to the new axes or system.

85. *To change the origin to the point  $(h, k)$  without changing the direction of the axes.*

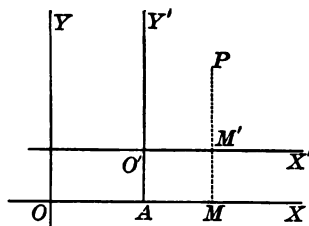


Fig. 41.

Let  $OX, OY$  be the old axes,  $O'X', O'Y'$  the new; and let  $(x, y), (x', y')$  be the co-ordinates of the *same* point  $P$ , referred to the old and new axes respectively.

Then (Fig. 41)

$$OA = h, AO' = k, OM = x, MP = y, O'M' = x', M'P = y'.$$

$$x = OA + AM = OA + O'M' = x' + h.$$

$$y = MM' + M'P = AO' + M'P = y' + k.$$

These relations are equally true for rectangular and oblique co-ordinates.

Hence, to find what the equation of a curve becomes when the origin is transferred to a point  $(h, k)$ , the new axes running parallel to the old, we must substitute for  $x$  and  $y$  the values given above.

After the substitution, we may, of course, write  $x$  and  $y$  instead of  $x'$  and  $y'$ ; so that practically the change is effected by simply writing  $x + h$  in place of  $x$ ,  $y + k$  in place of  $y$ .

If, however, we wish to transform a point  $(x, y)$  from the new to the old system, we must write  $x - h$  in place of  $x$  and  $y - k$  in place of  $y$ .

86. To change the reference of a curve from one set of rectangular axes to another, the origin remaining the same.

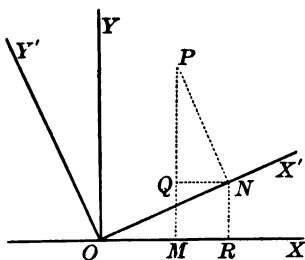


Fig. 42.

Let  $(x, y)$  be a point  $P$  referred to the old axes  $OX, OY$ ;  $(x', y')$ , the same point referred to the new axes  $OX', OY'$  (Fig. 42). Then

$$OM = x, \quad MP = y, \quad ON = x', \quad NP = y'.$$

Let the angle  $XOX' = \theta$ . Draw  $NQ, NR \perp$  to  $PM, OX$ , respectively; then

$$NPQ = QNO = RON = \theta.$$

Hence  $OM = OR - RM = OR - NQ = ON \cos \theta - PN \sin \theta$ .

Or  $x = x' \cos \theta - y' \sin \theta$ .

And  $PM = MQ + QP = RN + QP = ON \sin \theta + PN \cos \theta$ .

Or  $y = x' \sin \theta + y' \cos \theta$ .

Therefore, to find what the equation of a curve becomes when referred to the new axes, we must write

$$x \cos \theta - y \sin \theta \text{ for } x, \quad x \sin \theta + y \cos \theta \text{ for } y.$$

87. *To change the reference of a curve from one set of rectangular axes to another, both the origin and the direction of the axes being changed.*

First transform the equation to axes through the new origin, parallel to the old axes. Then turn these axes through the required angle.

If  $(h, k)$  is the new origin referred to the old axes,  $\theta$  the angle between the old and new axes of  $x$ , we obtain as the values of  $x$  and  $y$  for any point  $P$ , in terms of the new co-ordinates,

$$\begin{aligned} x &= h + x' \cos \theta - y' \sin \theta, \\ y &= k + x' \sin \theta + y' \cos \theta. \end{aligned}$$

In making all these transformations, attention must be paid to the *signs* of  $h$ ,  $k$ , and  $\theta$ .

88. *To change the reference of a curve from rectangular to oblique axes, the origin remaining the same.*

Let  $\alpha, \beta$  be the angles formed by the *positive* directions of the new axes  $OX', OY'$  (Fig. 43) with the positive direction of  $OX$ . Let the old co-ordinates of a point  $P$  be  $x, y$ ; and the new co-ordinates,  $x', y'$ . Then from the right triangles  $ORN, PQN$  we readily obtain the formulas

$$\begin{aligned} x &= x' \cos \alpha + y' \cos \beta, \\ y &= x' \sin \alpha + y' \sin \beta. \end{aligned}$$

Investigate the special case when  $\beta = \alpha + 90^\circ$ .



89. To deduce the formulas for finding the polar equation of a curve from its rectangular equation.

Let  $x, y$  be the rectangular co-ordinates of any point  $P$ , and  $\rho, \theta$  its polar co-ordinates.

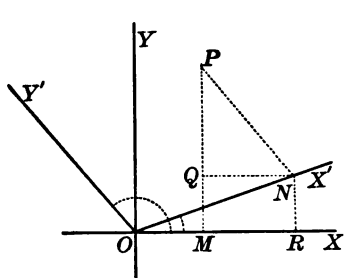


Fig. 43.

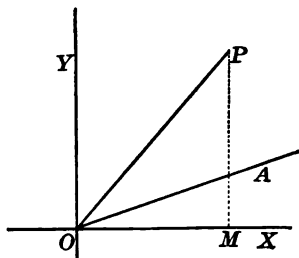


Fig. 44.

(i.) Let the origin of rectangular co-ordinates be the pole, and let the polar axis coincide with the axis of  $x$ .

Then (Fig. 44)  $OM = OP \cos MOP$ ,  
 $PM = OP \sin MOP$ .

Or  $x = \rho \cos \theta$ ,  
 $y = \rho \sin \theta$ .

(ii.) If the pole is the point  $(h, k)$ , we have

$$x = h + \rho \cos \theta,$$

$$y = k + \rho \sin \theta.$$

(iii.) If the pole coincides with the origin, but the polar axis  $OA$  makes the angle  $\alpha$  with the axis of  $x$ , we obtain

$$x = \rho \cos (\theta + \alpha),$$

$$y = \rho \sin (\theta + \alpha).$$

(iv.) If the pole is the point  $(h, k)$ , and the polar axis makes the angle  $\alpha$  with the axis of  $x$ ,

$$x = h + \rho \cos (\theta + \alpha),$$

$$y = k + \rho \sin (\theta + \alpha).$$

90. *To deduce the formulas for finding the rectangular equation of a curve from its polar equation.*

From the results in cases (i.) and (ii.) of § 89 (the only cases of importance), we readily obtain

$$\text{In case (i.), } \rho^2 = x^2 + y^2, \quad \tan \theta = \frac{y}{x}.$$

$$\text{In case (ii.), } \rho^2 = (x - h)^2 + (y - k)^2, \quad \tan \theta = \frac{y - k}{x - h}.$$

91. *The degree of an equation is not altered by passing from one set of axes to another.*

For, however the axes may be changed, the new equation is always obtained by substituting for  $x$  and  $y$  expressions of the form

$$ax + by + c \text{ and } a'x + b'y + c'.$$

These expressions are of the first degree, and therefore, if they replace  $x$  and  $y$  in the equation, the degree of the equation cannot be *raised*. Neither can it be *lowered*; for if it could be lowered, it might be raised by returning to the original axes, and therefore to the original equation.

#### Ex. 26.

1. What does the equation  $y^2 - 4x + 4y + 8 = 0$  become when the origin is changed to the point  $(1, -2)$ ?

Transform the equation of the circle  $(x - a)^2 + (y - b)^2 = r^2$  by changing the origin

2. To the centre of the circle.

3. To the left-hand end of the horizontal diameter.

4. To the upper end of the vertical diameter.

5. What does the equation  $x^2 + y^2 = r^2$  become if the axes are turned through the angle  $\alpha$ ?

6. What does the equation  $x^2 - y^2 = a^2$  become if the axes are turned through  $-45^\circ$ ?

7. The equation of a curve referred to rectangular axes is  $x - xy - y = 0$ . Transform it to new axes, whose origin is the point  $(-1, 1)$ , the new axis of  $y$  bisecting two of the angles formed by the old axes.

8. Change the following equations to polar co-ordinates, taking the pole at the origin and the polar axis to coincide with the axis of  $x$ :

$$(i.) x^2 + y^2 = a^2. \quad (ii.) x^2 - y^2 = a^2.$$

9. Change the equation  $x^2 = 4ay$  to polar co-ordinates, (i.) taking the pole at the origin; (ii.) taking the pole at the point  $(a, 0)$ .

10. Change the following equations to rectangular co-ordinates, the origin coinciding with the pole, and the polar axis with the axis of  $x$ :

$$(i.) \rho = a, \quad (ii.) \rho = a \cos \theta, \quad (iii.) \rho^2 \cos 2\theta = a^2.$$

Transform the following equations by changing the origin to the point given as a new origin:

$$11. x + y + 2 = 0; \text{ the new origin } (-2, 0).$$

$$12. 2x - 5y - 10 = 0; \text{ the new origin } (5, -2).$$

$$13. 3x^2 + 4xy + y^2 - 5x - 6y - 3 = 0; \text{ new origin } (\frac{7}{2}, -4).$$

$$14. x^2 + y^2 - 2x - 4y = 20; \text{ new origin } (1, 2).$$

$$15. x^2 - 6xy + y^2 - 6x + 2y + 1 = 0; \text{ new origin } (0, -1).$$

16. Transform the equation  $x^2 - y^2 + 6 = 0$  by turning the axes through  $45^\circ$ .

17. Transform the equation  $(x + y - 2a)^2 = 4xy$  by turning the axes through  $45^\circ$ .

18. Transform the equation  $9x^2 - 16y^2 = 144$  to oblique axes, such that the new axis of  $x$  makes with the old axis of  $x$  a negative angle whose tangent  $= -\frac{3}{4}$ ; and the new axis of  $y$  makes with the old axis of  $x$  a positive angle whose tangent is  $\frac{3}{4}$ .

## Ex. 27. (Review.)

1. Find the distance from the point  $(-2b, b)$  to the origin, the axes making the angle  $60^\circ$ .

2. The axes making the angle  $\omega$ , find the distance from the point  $(1, -1)$  to the point  $(-1, 1)$ .

3. The axes making the angle  $\omega$ , find the distance from the point  $(0, 2)$  to the point  $(3, 0)$ .

Determine the distance between the following points given by polar co-ordinates:

4.  $(a, \theta)$  and  $(b, \phi)$ .

5.  $(a, \theta)$  and  $(a, -\theta)$ .

6.  $(a, \theta)$  and  $(-a, -\theta)$ .

7.  $(2a, 30^\circ)$  and  $(a, 60^\circ)$ .

8. Show that the polar co-ordinates  $(\rho, \theta)$ ,  $(-\rho, \pi + \theta)$ ,  $(-\rho, \theta - \pi)$  all represent the same point.

9. Transform the equation  $8x^2 + 8xy + 4y^2 + 12x + 8y + 1 = 0$  to the new origin  $(-\frac{1}{2}, -\frac{1}{2})$ .

10. Transform the equation  $6x^2 + 3y^2 - 24x + 6 = 0$  to the new origin  $(2, 0)$ .

11. Transform the equation  $\frac{x}{a} + \frac{y}{b} = 1$  by changing the origin to the point  $(\frac{a}{2}, \frac{b}{2})$  and turning the axes through an angle  $\phi$ , such that  $\tan \phi = -\frac{b}{a}$ .

12. Transform the equation  $17x^2 - 16xy + 17y^2 = 225$  to axes which bisect the axes of the old system.

Transform the following rectangular equations to polar equations, the polar axis in each case coinciding with, or being parallel to, the axis of  $x$ , and the pole being at the point whose co-ordinates are given :

13.  $x^2 + y^2 = 8ax$ ; the pole  $(0, 0)$ .
14.  $x^2 + y^2 = 8ax$ ; the pole  $(4a, 0)$ .
15.  $y^2 - 6y - 5x + 9 = 0$ ; the pole  $(\frac{1}{2}, 3)$ .
16.  $x^2 - y^2 - 4x - 6y - 54 = 0$ ; the pole  $(2, -3)$ .
17.  $(x^2 + y^2)^2 = k^2(x^2 - y^2)$ ; the pole  $(0, 0)$ .

Transform the following polar equations to rectangular axes, the origin being at the pole and the axis of  $x$  coinciding with the polar axis :

18.  $\rho^2 \sin 2\theta = 2a^2$ .
19.  $\rho = k \sin 2\theta$ .
20.  $\rho(\sin 3\theta + \cos 3\theta) = 5k \sin \theta \cos \theta$ .

21. Through what angle must a set of rectangular axes be turned in order that the new axis of  $x$  may pass through the point  $(5, 7)$ ?

22. The rectangular equation of a straight line is  $Ax + By + C = 0$ . Through what angle must the axes be turned in order

- (i.) that the term containing  $x$  may disappear?
- (ii.) that the term containing  $y$  may disappear?

23. Deduce the following formulas for changing from one set of oblique axes to another, the origin remaining the same :

$$x = \frac{x' \sin(\omega - \alpha)}{\sin \omega} + \frac{y' \sin(\omega - \beta)}{\sin \omega},$$

$$y = \frac{x' \sin \alpha}{\sin \omega} + \frac{y' \sin \beta}{\sin \omega}.$$

NOTE. In these formulas  $\omega$  denotes the angle formed by the old axes,  $\alpha$  and  $\beta$  those formed by the positive directions of the new axes with the positive direction of the old axis of  $x$ .

24. From the formulas of No. 23 deduce those of § 88.

## CHAPTER V.

### THE PARABOLA.

#### THE EQUATION OF THE PARABOLA.

**92.** A **Parabola** is the locus of a point whose distance from a fixed point is always equal to its distance from a fixed straight line.

The fixed point is called the **Focus**; the fixed straight line, the **Directrix**.

The straight line through the focus, and perpendicular to the directrix, is called the **Axis** of the parabola.

The intersection of the axis and the directrix is called the **Foot** of the axis.

The point in the axis half way between the focus and the directrix is, from the definition, a point of the curve; this point is called the **Vertex** of the parabola.

The straight line joining any point of the curve to the focus is called the **Focal Radius** of the point.

A straight line passing through the focus and limited by the curve is called a **Focal Chord**.

The focal chord perpendicular to the axis is called the **Latus Rectum** or **Parameter**.

**93.** *To construct a parabola, having given the focus and the directrix.*

**I. By Points.** Let  $F$  (Fig. 45) be the focus,  $CE$  the directrix. Draw the axis  $FD$ , and bisect  $FD$  in  $A$ ; then  $A$  is the vertex of the curve. At any point  $M$  in the axis erect a perpendicular. From  $F$  as centre, with  $DM$  as radius, cut this

perpendicular in  $P$  and  $Q$ ; then  $P$  and  $Q$  are two points of the curve, for  $FP = DM =$  distance of  $P$  or  $Q$  from  $CE$ . In the same way we can find as many points of the curve as we please. After a sufficient number of points has been found, we draw a continuous curve through them.

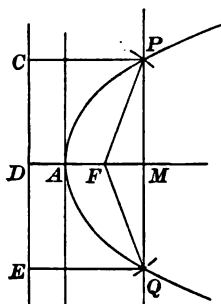


Fig. 45.

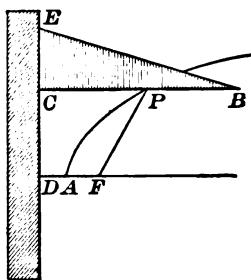


Fig. 46.

II. *By Motion.* Place a ruler so that one of its edges shall coincide with the directrix  $DE$  (Fig. 46). Then place a triangular ruler  $BCE$  with the edge  $CE$  against the edge of the first ruler. Take a string whose length is equal to  $BC$ ; fasten one end at  $B$  and the other end at  $F$ . Then slide the ruler  $BCE$  along the directrix, keeping the string tightly pressed against the ruler by the point of a pencil  $P$ . The point  $P$  will trace a parabola; for during the motion we always have  $PF = PC$ .

94. *To find the rectangular equation of the parabola, when its axis is taken as the axis of  $x$  and its vertex as the origin.*

Let  $F$  (Fig. 45) be the focus,  $CE$  the directrix,  $DFX$  the axis,  $A$  the vertex and origin; also let  $2p$  denote the known distance  $FD$ .

Let  $P$  be any point of the curve; then its co-ordinates are

$$AM = x, \quad PM = y.$$

Draw  $PC \perp$  to  $CE$ ; then by the definition of the curve

$$FP = PC = DM.$$

Therefore  $\overline{FP}^2 = \overline{DM}^2.$

Now  $\overline{FP}^2 = \overline{PM}^2 + \overline{FM}^2 = y^2 + (x - p)^2,$

and  $\overline{DM}^2 = (x + p)^2.$

Therefore  $y^2 + (x - p)^2 = (x + p)^2.$

Whence  $y^2 = 4px.$  [26]

This is called the *principal* equation of a parabola.

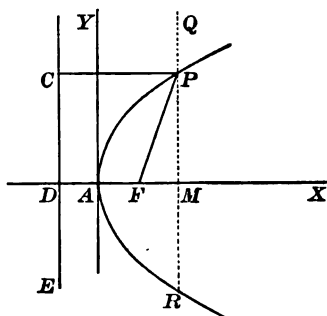


Fig. 47.

95. Since  $y^2$  and  $p$  in equation [26] are positive,  $x$  must always be positive; therefore the curve lies wholly on the positive side of the axis of  $y$ .

A further examination of equation [26] shows that the curve, (i.) passes through the origin, (ii.) is symmetrical with respect to the axis of  $x$ , (iii.) extends towards the right without limit, and (iv.) recedes from the axis of  $x$  without limit.

96. Any point  $(h, k)$  is outside, on, or inside the parabola  $y^2 = 4px$ , according as  $k^2 - 4ph$  is positive, zero, or negative.

Let  $Q$  be the point  $(h, k)$ , and let its ordinate meet the curve in  $P$ .



If  $k^2 - 4ph = 0$ , the point  $(h, k)$  satisfies equation [26], and therefore  $Q$  coincides with  $P$ .

If  $k^2 - 4ph$  is positive, or  $k^2 > 4ph$ , then, since  $\overline{PM}^2 = 4ph$ , we have  $\overline{QM}^2 > \overline{PM}^2$ , or  $QM > PM$ ; hence  $Q$  is outside the curve.

If  $k^2 - 4ph$  is negative, we may prove similarly that  $Q$  must be inside the curve.

97. For  $x = p$ ,  $y = \pm 2p$ . But these two values of  $y$  make up the latus rectum. Hence the *latus rectum*  $= 4p$ .

COR. From the equation  $y^2 = 4px$ , it follows that

$$x : y = y : 4p;$$

that is, *the latus rectum is a third proportional to any abscissa and its corresponding ordinate.*

98. If  $(x_1, y_1)$  and  $(x_2, y_2)$  be any two points on the parabola, we have

$$y_1^2 = 4px_1, \quad y_2^2 = 4px_2.$$

Hence, 
$$y_1^2 : y_2^2 = x_1 : x_2;$$

that is, *the squares of the ordinates of any two points on the parabola are to each other as their abscissas.*

99. To find the points in which the straight line  $y = mx + c$  meets the parabola  $y^2 = 4px$ .

Regarding these equations as simultaneous, and eliminating  $x$ , we have

$$y^2 = 4p \frac{y - c}{m}. \quad (1)$$

Whence 
$$y = \frac{2p}{m} \pm \frac{2p}{m} \sqrt{\frac{p - mc}{p}}. \quad (2)$$

From (2) it follows that  $y = mx + c$  has two distinct, two coincident, or no points in common with  $y^2 = 4px$ , according as  $p - mc >, =, \text{ or } < 0$ .

COR. If  $p - mc = 0$ , or  $c = p + m$ ,  $y = mx + c$  will be a tangent; that is,

$$y = mx + \frac{p}{m} \quad (3)$$

is a tangent to  $y^2 = 4px$  in terms of its slope.

## Ex. 28.

1. Show that the distance of any point of the parabola  $y^2 = 4px$  from the focus is equal to  $p + x$ .

2. Find the equation of a parabola, taking as axes the axis of the curve and the directrix.

3. Find the equation of a parabola, taking the axis of the curve as the axis of  $x$  and the focus as the origin.

4. The distance from the focus of a parabola to the directrix = 5. Write its equation,

(i.) If the origin is taken at the vertex.

(ii.) If the origin is taken at the focus.

(iii.) If the axis and directrix are taken as axes.

5. The distance from the focus to the vertex of a parabola is 4. Write its equations for the three cases enumerated in No. 4.

6. For what point of the parabola  $y^2 = 18x$  is the ordinate equal to three times the abscissa?

7. Find the latus rectum for the following parabolas:

$$y^2 = 6x, \quad y^2 = 15x, \quad by^2 = ax.$$

Find the points common to the following parabolas and straight lines:

8.  $y^2 = 9x, \quad 3x - 7y + 30 = 0.$

9.  $y^2 = 3x, \quad x - 4y + 12 = 0.$

10.  $y^2 = 4x, \quad x = 9, \quad x = 0, \quad x = -2.$

11.  $y^2 = 4x, \quad y = 6, \quad y = -8.$

12. What must be the value of  $p$  in order that the parabola  $y^2 = 4px$  may pass through the point  $(9, -12)$ ?

13. For what point of the parabola  $y^2 = 32x$  is the ordinate equal to 4 times the abscissa?

14. The equation of a parabola is  $y^2 = 8x$ . What is the equation of (i.) its axis, (ii.) its directrix, (iii.) its latus rectum, (iv.) a focal chord through the point whose abscissa = 8, (v.) a chord passing through the vertex and the negative end of the latus rectum?

15. The equation of a parabola is  $y^2 = 16x$ . Find the equation of (i.) a chord through the points whose abscissas are 4 and 9, and ordinates positive; (ii.) the circle passing through the vertex and the ends of the latus rectum.

16. If the distance of a point from the focus of the parabola  $y^2 = 4px$  is equal to the latus rectum, what is the abscissa of the point?

17. In the parabola  $y^2 = 4px$  an equilateral triangle is inscribed so that one vertex is at the origin. What is the length of one of its sides?

18. A double ordinate of a parabola  $= 8p$ . Prove that straight lines drawn from its ends to the vertex are perpendicular to each other.

Explain how to construct a parabola, having given

19. The directrix and the vertex.

20. The focus and the vertex.

21. The axis, vertex, and latus rectum.

22. The axis, vertex, and a point of the curve.

23. The axis, focus, and latus rectum.

24. Determine, as regards size and position, the relations of the following parabolas:

(i.)  $y^2 = 4px$ , (ii.)  $y^2 = -4px$ , (iii.)  $x^2 = 4py$ , (iv.)  $x^2 = -4py$ .

## TANGENTS AND NORMALS.

100. To find the equation of the tangent and of the normal to the parabola  $y^2 = 4px$  at any point  $(x_1, y_1)$ .

Let  $(x_1, y_1), (x_2, y_2)$  be any two points on the parabola; then the equation of the secant through them is

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}. \quad (1)$$

Since  $(x_1, y_1)$  and  $(x_2, y_2)$  are on the curve  $y^2 = 4px$ , we have

$$y_1^2 = 4px_1, \quad y_2^2 = 4px_2.$$

Whence 
$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{4p}{y_2 + y_1}.$$

By substituting in (1), the equation of the secant becomes

$$\frac{y - y_1}{x - x_1} = \frac{4p}{y_2 + y_1}. \quad (2)$$

Now if  $(x_2, y_2)$  be made to coincide with  $(x_1, y_1)$ , (2) becomes the equation of the tangent at  $(x_1, y_1)$ . Putting  $y_2 = y_1$ , clearing of fractions, and remembering that  $y_1^2 = 4px_1$ , we obtain as the equation of the tangent at  $(x_1, y_1)$ ,

$$y_1 y = 2p(x + x_1). \quad [27]$$

The normal passes through  $(x_1, y_1)$ , and is perpendicular to the tangent; hence its equation is, by [27] and § 46,

$$y - y_1 = -\frac{y_1}{2p}(x - x_1). \quad [28]$$

101. If in [27] we put  $y = 0$ , we obtain

$$x = -x_1, \text{ or } TA = AM \text{ (Fig. 48).}$$

Therefore the subtangent is bisected at the vertex.

If in [28] we put  $y = 0$ , we obtain

$$x = x_1 + 2p, \text{ or } x - x_1 [= MN] = 2p \text{ (Fig. 48).}$$

Hence, the subnormal is constant and equal to the semi-latus rectum.

COR. These properties furnish simple methods for drawing tangents to the parabola. Thus, to draw a tangent to the parabola at  $P$  (Fig. 48), draw the ordinate  $PM$ , lay off  $AT = AM$ , and draw  $PT$ , which will be the tangent at  $P$  by § 101. Or lay off  $MN = FD$ , and draw  $PN$ ; then  $PT$  perpendicular to  $PN$  at  $P$  will be the tangent at  $P$ .

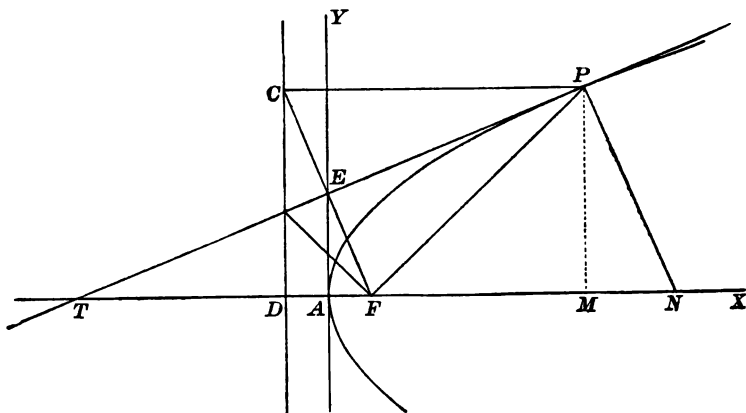


Fig. 48.

102. In the triangle  $FPT$  (Fig. 48) we have

$$FT = FA + AT = p + x,$$

$$FP = PC = DM = DA + AM = p + x.$$

Therefore  $FT = FP$ .

Hence the angle

$$FPT = PTF = TPC, \text{ or}$$

*The tangent to a parabola at any point makes equal angles with the axis of the curve and the focal radius to the point of contact.*

## EX. 29.

1. The normal to a parabola at any point bisects the angle between the focal radius and the line drawn through the point parallel to the axis.

NOTE. The use of parabolic reflectors depends on this property. A ray of light issuing from the focus and falling on the reflector is reflected in a line parallel to the axis of the reflector.

2. Explain how to draw a tangent and a normal to a given parabola at a given point.

3. Prove that  $FC$  (Fig. 48) is perpendicular to  $PT$ .

4. Prove that the tangent  $y = mx + \frac{p}{m}$  touches the parabola  $y^2 = 4px$  at the point  $\left(\frac{p}{m^2}, \frac{2p}{m}\right)$ .

5. Prove that the equation of a normal to the parabola  $y^2 = 4px$  in terms of its slope is  $y = mx - mp(2 + m^2)$ .

6. What are the equations of a tangent and a normal to the parabola  $y^2 = 5x$ , passing through the point whose abscissa is 20 and ordinate positive?

7. What are the equations of the tangents and the normals to the parabola  $y^2 = 12x$ , drawn through the ends of the latus rectum? Find the area of the figure which they enclose.

8. Given the parabola  $y^2 = 10x$ . Through the point whose abscissa is 7 and ordinate positive a tangent and a normal are drawn. Find the lengths of the tangent, the normal, the subtangent, and the subnormal.

9. A tangent to the parabola  $y^2 = 20x$  makes with the axis of  $x$  an angle of  $45^\circ$ . Determine the point of contact.

10. Show that the focus  $F$  (Fig. 48) is equidistant from the points  $P, T, N$ . What easy way of drawing a tangent and a normal is suggested by this theorem?

11. If  $F$  is the focus of a parabola, and  $Q, R$  denote the points in which a tangent cuts the directrix and the latus rectum produced, prove that  $FQ = FR$ .

12. Prove that tangents drawn through the ends of the latus rectum are  $\perp$  to each other.

13. Find the distances of the vertex and the focus from the tangent  $y = mx + \frac{p}{m}$ .

14. The points of contact of two tangents are  $(x_1, y_1)$  and  $(x_2, y_2)$ . Find their point of intersection.

15. A tangent to the parabola  $y^2 = 4px$  cuts equal intercepts on the axes. What is its equation? What is the point of contact? What is the value of the intercept?

16. Through what point in the axis of  $x$  must tangents to the parabola  $y^2 = 4px$  be drawn in order that they may form with the tangent, through the vertex, an equilateral triangle?

17. For what point of the parabola  $y^2 = 4px$  is the normal equal to twice the subtangent?

18. For what point of the parabola  $y^2 = 4px$  is the normal equal to the difference between the subtangent and the subnormal?

19. Find the equation of a tangent to the parabola  $y^2 = 5x$  parallel to the straight line  $3x - 2y + 7 = 0$ . Also find the point of contact.

20. Find the equation of the straight line which touches the parabola  $y^2 = 12x$  and makes an angle of  $45^\circ$  with the line  $y = 3x - 4$ . Also find the point of contact.

21. Find the equation of a straight line which touches the parabola  $y^2 = 16x$  and passes through the point  $(-4, 8)$ .

22. If a normal to a parabola meet the curve again in the point  $Q$ , find the length of  $PQ$ .

23. Prove by the secant method that the equation of a tangent to the parabola  $y^2 = 4px - 4p^2$ , at the point  $(x_1, y_1)$  is

$$y_1 y = 2p(x + x_1) - 4p^2.$$

24. Find the equations of the tangents and normals to the parabola  $y^2 - 8x - 6y - 63 = 0$ , drawn through the points whose common abscissa  $= -1$ .

25. What are the equations of tangents to the following parabolas:

$$(i.) y^2 = -4px? \quad (ii.) x^2 = 4py? \quad (iii.) x^2 = -4py?$$

### Ex. 30. (Review.)

NOTE. If not otherwise specified, the axis of the parabola and the tangent at the vertex are to be assumed as axes of co-ordinates.

What is the equation of a parabola,

1. If the axis and directrix are taken as axes, and the focus is the point  $(12, 0)$ ?

2. If the axis and tangent at the vertex are the two axes, and  $(25, 20)$  is a point on the curve?

3. If the same axes are taken, and the focus is the point  $(-4\frac{1}{2}, 0)$ ?

4. If the axis is parallel to the axis of  $x$ , the vertex is the point  $(5, -3)$ , and the latus rectum  $= 5\frac{1}{2}$ ?

5. If the axis is the line  $y = -7$ , the abscissa of the vertex  $= 3$ , and one point is  $(4, -5)$ ?

6. If the curve passes through the points  $(0, 0)$ ,  $(3, 2)$ ,  $(3, -2)$ ?

7. If the curve passes through the points  $(0, 0)$ ,  $(3, 2)$ ,  $(-3, 2)$ ?



8. What is the latus rectum of the parabola  $2y^2 = 3x$ ? What is the equation of its directrix, and of the focal chords passing through the points whose abscissa = 6?

9. Describe the change of form which the parabola  $y^2 = 4px$  undergoes as we suppose  $p$  to diminish without limit.

10. Find the intercepts of the parabola  $y^2 + 4x - 6y - 16 = 0$ .

11. One vertex of an equilateral triangle coincides with the focus, and the others lie in the parabola  $y^2 = 4px$ . Find the length of one side.

12. The latus rectum of a parabola = 8; find

(i.) Equation of a tangent through its positive end.

(ii.) Distance from the focus to this tangent.

(iii.) Equation of the normal at this point.

13. What is the equation of the chord passing through the two points of the parabola  $y^2 = 8x$  for which  $x_1 = 2$ ,  $y_1 > 0$ , and  $x_2 = 18$ ,  $y_2 < 0$ ?

14. Find the equation of the chord of the parabola  $y^2 = 4px$  which is bisected at a given point  $(x_1, y_1)$ .

15. In what points does the line  $x + y = 12$  meet the parabola  $y^2 + 2x - 12y + 16 = 0$ ?

16. In what points does the line  $3y = 2x + 8$  meet the parabola  $y^2 - 4x - 8y + 24 = 0$ ?

17. Find the equations of tangents from the origin to the parabola  $(y - b)^2 = 4p(x - a)$ .

18. Describe the position of the parabola  $y^2 + 2x + 4 = 0$  with respect to the axes, and determine its latus rectum, vertex, focus, and directrix.

19. What is the distance from the origin to a normal drawn through the end of the latus rectum of the parabola

$$y^2 = 4a(x - a)?$$

Find the equation of a parabola,

20. If the equation of a tangent is  $4y = 3x - 12$ .

21. If a focal radius = 10, and its equation is  $3y = 4x - 8$ .

22. If for a point of the curve the focal radius =  $r$ , the length of the tangent =  $t$ .

23. If for a point of the curve the focal radius =  $r$ , the length of the normal =  $n$ .

24. If for a point of the curve the length of the tangent =  $t$ , the length of the normal =  $n$ .

25. If for a point of the curve the focal radius =  $r$ , the subtangent =  $s$ .

26. Two parabolas have the same vertex, and the same latus rectum  $4p$ , but their axes are  $\perp$  to each other. What is the length of their common chord?

27. Through the three points of the parabola  $y^2 = 12x$ , whose ordinates are 2, 3, 6, tangents are drawn. Show that the circle circumscribed about the triangle formed by the tangents passes through the focus.

28. A tangent to the parabola  $y^2 = 4px$  makes the angle  $30^\circ$  with the axis of  $x$ . At what point does it cut the axis?

29. For what point of the parabola  $y^2 = 4px$  is the length of the tangent equal to 4 times the abscissa of the point of contact?

30. The product of the tangent and normal is equal to twice the square of the ordinate of the point of contact. Find the point of contact and the inclination of the tangent to the axis of  $x$ .

31. Two tangents to a parabola are perpendicular to each other. Find the product of their subtangents.

32. Prove that the circle described on a focal radius as diameter touches the tangent drawn through the vertex.

33. Prove that the circle described on a focal chord as diameter touches the directrix.

Find the locus of the middle points

34. Of all the ordinates of a parabola.

35. Of all the focal radii.

36. Of all the focal chords.

37. Of all chords passing through the vertex.

38. Of all chords that meet at the foot of the axis.

Two tangents to the parabola  $y^2 = 4px$  make the angles  $\theta, \theta'$  with the axis of  $x$ ; find the locus of their intersection

39. If  $\cot \theta + \cot \theta' = k$ .

41. If  $\tan \theta \tan \theta' = k$ .

40. If  $\cot \theta - \cot \theta' = k$ .

42. If  $\sin \theta \sin \theta' = k$ .

43. Find the locus of the centre of a circle which passes through a given point and touches a given straight line.

### SUPPLEMENTARY PROPOSITIONS.

**103.** *Two distinct, two coincident, or no real tangents can be drawn to a parabola from any point  $(h, k)$ , according as the point is without, on, or within the curve.*

Let the tangent  $y = mx + \frac{p}{m}$  pass through the points  $(h, k)$ ;  
then

$$k = mh + \frac{p}{m},$$

or

$$hm^2 - km + p = 0.$$

Whence

$$m = \frac{k \pm \sqrt{k^2 - 4ph}}{2h}.$$

These values of  $m$  are real and unequal, equal, or imaginary, according as  $k^2 - 4ph >, =, \text{ or } < 0$ ; that is, according as  $(h, k)$  is without, on, or within the parabola; hence the proposition.

**104.** *To find the equation of the chord of contact of two tangents drawn from any external point  $(h, k)$  to the parabola  $y^2 = 4px$ .*

Let  $(x_1, y_1)$  and  $(x_2, y_2)$  be the points of contact; then the equations of the tangents are

$$y_1y = 2p(x + x_1),$$

$$y_2y = 2p(x + x_2).$$

Since  $(h, k)$  is in both these lines, we have

$$ky_1 = 2p(x_1 + h), \quad (1)$$

$$ky_2 = 2p(x_2 + h). \quad (2)$$

From equations (1) and (2) we see that both the points  $(x_1, y_1)$  and  $(x_2, y_2)$  lie in the straight line whose equation is

$$ky = 2p(x + h). \quad (3)$$

Hence (3) is the equation required.

**105.** *To find the equation of the polar of the pole  $(h, k)$  with regard to the parabola  $y^2 = 4px$ .*

Let  $P$  be the fixed point  $(h, k)$ ,  $PQR$  one position of the revolving chord, and let the tangents at  $Q$  and  $R$  intersect in  $P_1(x_1, y_1)$ ; it is required to find the locus of  $P_1$ , as the chord turns about  $P$ .

Since  $PR$  is the chord of contact of tangents drawn from the point  $P_1(x_1, y_1)$ , its equation is (§ 104)

$$y_1y = 2p(x + x_1). \quad (1)$$

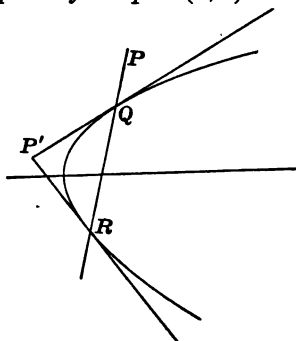


Fig. 49.

Since (1) passes through  $(h, k)$ , we have

$$y_1 k = 2p(h + x_1). \quad (2)$$

But  $(x_1, y_1)$  is any point on the required locus, and by (2) its co-ordinates satisfy the equation

$$ky = 2p(x + h). \quad (3)$$

Hence (3) is the required equation, and the polar is a straight line.

COR. When the pole  $(h, k)$  is on the curve, the polar is evidently a tangent at  $(h, k)$ ; when the pole  $(h, k)$  is without the curve, the polar is the chord of contact of tangents from  $(h, k)$ . Thus the *tangent* and *chord of contact* are particular cases of the *polar*.

Proposition 74 holds true for poles and polars with respect to a parabola, and may be proved as in the case of the circle.

106. To find the locus of the middle points of parallel chords in the parabola  $y^2 = 4px$ .

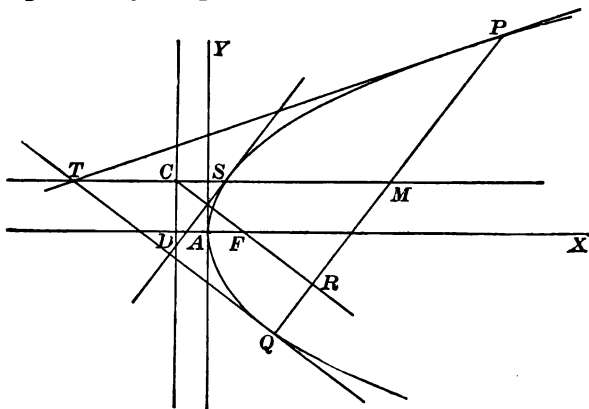


Fig. 50.

Let the equation of any one of the chords  $PQ$  (Fig. 50) be  $y = mx + c$ , and let it meet the curve in the points  $(x_1, y_1)$ ,  $(x_2, y_2)$ .

Then (§ 100) 
$$m = \frac{4p}{y_1 + y_2}. \quad (1)$$

Let  $(x, y)$  be its middle point  $M$ ; then  $2y = y_1 + y_2$ . By substitution in (1) we obtain

$$m = \frac{2p}{y}, \text{ or } y = \frac{2p}{m}, \quad (2)$$

a relation which holds true for all parallel chords, because  $m$  is the same for all the chords. The required locus, therefore, is represented by (2), and is a straight line parallel to the axis of  $x$ .

Therefore every diameter of a parabola is a straight line parallel to its axis.

Conversely, every straight line parallel to the axis is a diameter; for  $m$ , and therefore  $\frac{2p}{m}$ , may have any value whatever.

107. Let the diameter through  $M$  meet the curve at  $S$ , and conceive the straight line  $PQ$  to move parallel to itself till  $P$  and  $Q$  coincide at  $S$ ; then the straight line becomes the tangent at  $S$ ; therefore

*The tangent drawn through the extremity of a diameter is parallel to the chords of that diameter.*

108. From the focus  $F$  draw  $FC \perp$  to  $PQ$ , and let  $FC$  meet the directrix in the point  $C$ . If  $\theta$  denote the angle which the chord  $PQ$  makes with the axis of  $x$ , it easily follows that  $DCF = \theta$ ; then we have

$$CD = FD \cot \theta = \frac{2p}{m};$$

hence, by (2), § 106,

*The perpendicular from the focus to a chord meets the diameter of the chord in the directrix.*

Moreover, since  $DS$  (Fig. 50) is parallel to  $QP$ , the perpendicular from the focus to a tangent meets the diameter through the point of contact in the directrix.

109. Let the tangents drawn through  $P$  and  $Q$  meet in the point  $T$ . Regarding their equations,

$$y_1 y = 2p(x + x_1),$$

$$y_2 y = 2p(x + x_2),$$

as simultaneous, we obtain for the value of the ordinate of  $T$

$$y = \frac{2p(x_2 - x_1)}{y_2 - y_1} = \frac{2p}{m}. \quad \text{Hence}$$

*Tangents drawn through the ends of a chord meet in the diameter of the chord.*

110. To find the locus of the foot of a perpendicular from the focus to a tangent.

Let the equation of the tangent be

$$y = mx + \frac{p}{m}.$$

Then the equation of the perpendicular will be

$$y = -\frac{x}{m} + \frac{p}{m}.$$

Since these two lines have the same intercept on the axis of  $y$ , they meet in that axis; hence the tangent through the vertex is the required locus.

111. Since  $FP = PC$  (Fig. 48) and angle  $EPC = EPF$  therefore the tangent  $PT$  is perpendicular to  $FC$  at its middle point, and every point in it is equally distant from  $F$  and  $C$ .

112. *Tangents at right angles intersect in the directrix.*

Let the equation of one tangent be

$$y = mx + \frac{p}{m}. \quad (1)$$

Then the equation of the other is

$$y = -\frac{x}{m} - mp. \quad (2)$$

Subtracting (2) from (1) we obtain for their common point

$$(x+p)\left(m+\frac{1}{m}\right)=0.$$

But  $m+\frac{1}{m}$  cannot be zero; hence  $x+p=0$ , or  $x=-p$ , which is the equation of the directrix.

113. The polar of the focus  $(p, 0)$  is

$$0=2p(x+p), \text{ or } x=-p.$$

Hence, *the polar of the focus is the directrix, or tangents at the ends of a focal chord intersect in the directrix.*

COR. From this result and § 112 it follows that *tangents through the ends of a focal chord intersect at right angles.*

114. *To find the equation of a parabola referred to any diameter and the tangent through its extremity as axes.*

Transform the equation  $y^2=4px$  to the diameter  $SX'$  (Fig. 51) and the tangent through  $S$  as new axes. Let  $m$  be the slope of the tangent,  $\theta$  the angle which the tangent makes with the diameter; then  $m=\tan \theta$ .

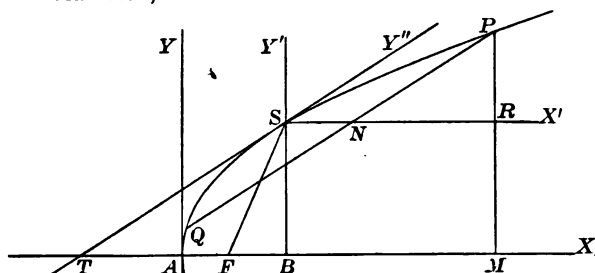


Fig. 51.

First transform to new parallax axes through  $S$ .

Now by § 106,  $BS=2p+m$ ; hence from  $y^2=2px$  we obtain  $AB=p+m^2$ . Therefore the new equation is

$$\left(y+\frac{2p}{m}\right)^2=4p\left(x+\frac{p}{m^2}\right),$$

or

$$my^2+4py=4pmx. \quad (1)$$



Now retain the axis of  $x$ , and turn the axis of  $y$  till it coincides with the tangent at  $S$ ; then for any point  $P$  we have

The old  $x = SR$ .                      The new  $x = SN$ .

The old  $y = RP$ .                      The new  $y = NP$ .

Now it is easily seen from Fig. 52 that

$$SR = SN + NP \cos \theta, \quad RP = NP \sin \theta.$$

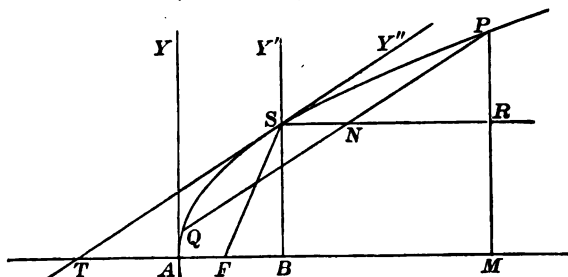


Fig. 52.

Therefore equation (1) is transformed to the new system by writing  $x + y \cos \theta$  in place of  $x$ , and  $y \sin \theta$  in place of  $y$ . Making this substitution, remembering that  $m = \tan \theta$ , and reducing, we obtain

$$y^2 = \frac{4p}{\sin^2 \theta} x, \quad (2)$$

an equation of the same form as  $y^2 = 4px$ .

Join  $S$  to the focus  $F$ , and denote  $FS$  by  $p'$ ; then

$$p' = p + AB = p + \frac{p}{m^2} = \frac{p(1 + m^2)}{m^2} = \frac{p}{\sin^2 \theta}.$$

Therefore equation (2) may be more simply written

$$y^2 = 4p'x, \quad (3)$$

where  $p'$  is the distance of the origin from the focus. It is easy to see that this equation includes the case where the axes are the axis of the curve and the tangent at the vertex.

The quantity  $4p'$  is called the **Parameter** of the diameter passing through  $S$ . When the diameter is the axis of the curve, it is called the **Principal Parameter**.

COR. Let the equation of a parabola referred to any diameter, and the tangent at the end of that diameter as axes, be  $y^2 = 4p'x$ . Since the investigations in §§ 99, 100 hold good whether the axes are at right angles or not, it follows immediately that the equation of the tangent at any point  $(x_1, y_1)$  is  $y_1y = 2p'(x + x_1)$ , and that the straight line  $y = mx + \frac{p'}{m}$  will touch the parabola for all values of  $m$ .

115. The latus rectum of a parabola = 16. What is the  $p$  of the curve if a diameter at the distance 12 from being the  $p$  and the tangent through its extremity, are taken as

Let  $P$  be the point on the curve; then the equation of that chord of the parabola  $\rho = FP =$  which is bisected at the point  $(h, k)$  is

$$k(y - k) = 2p(x - h).$$

$= 2p$ , that the parameter of any diameter is equal to  $\therefore \rho = \frac{1}{2}$ rd of that diameter.

Discussion. The axis is parallel to the axis of  $x$ ; and determine the latus rectum, the vertex, the focus, the axis, and the Since  $\cos \theta$

For  $\theta = 0$ , that in general the locus of  $y^2 + Ax + By + C = 0$

This shows that the axis of the parabola does not cut the curve to the right of the focus.

For  $\theta = \frac{1}{2}\pi$ ,  $\cos \theta = 0$ ;  $\therefore \rho = 2p =$  semi-latus rectum.

For  $\theta = \pi$ ,  $\cos \theta = -1$ ;  $\therefore \rho = p = FA$ .

For  $\theta = \frac{3}{2}\pi$ ,  $\cos \theta = 0$ ;  $\therefore \rho = 2p = FR'$ .

For  $\theta = 2\pi$ ,  $\cos \theta = 1$ ;  $\therefore \rho = \infty$ .

As  $\theta$  increases from zero to  $\pi$ ,  $\rho$  decreases from  $\infty$  to  $p$ . As  $\theta$  increases from  $\pi$  to  $2\pi$ ,  $\rho$  increases from  $p$  to  $\infty$ .



12. In the parabola  $y^2 = 13x$ , what is the equation of the ordinates of the diameter  $y + 11 = 0$ ?

13. In the parabola  $y^2 = 6x$ , what chord is bisected at the point  $(4, 3)$ ?

14. Given the parabola  $y^2 = 4px$ ; find the equation of the chord which passes through the vertex and is bisected by the diameter  $y = a$ . How can this chord be constructed?

15. The latus rectum of a parabola = 16. What is the equation of the curve if a diameter at the distance 12 from the focus, and the tangent through its extremity, are taken as axes?

16. Show that the equation of that chord of the parabola  $y^2 = 4px$  which is bisected at the point  $(h, k)$  is

$$k(y - k) = 2p(x - h).$$

17. Prove that the parameter of any diameter is equal to the focal chord of that diameter.

18. Prove that the locus of  $y^2 - 8y - 6x + 28 = 0$  is a parabola whose axis is parallel to the axis of  $x$ ; and determine the latus rectum, the vertex, the focus, the axis, and the directrix.

19. Prove that in general the locus of  $y^2 + Ax + By + C = 0$  is a parabola whose axis is parallel to the axis of  $x$ ; and determine its latus rectum, vertex, and axis.

20. Prove that in general the locus of  $x^2 + Ax + By + C = 0$  is a parabola whose axis is parallel to the axis of  $y$ ; and determine its latus rectum, vertex, and axis.

21. Find the locus of the centre of a circle which touches a given circle and also a given straight line.

22. The area and base of a triangle being given, find the locus of the intersection of perpendiculars dropped from the ends of the base to the opposite sides.

## CHAPTER VI.

### THE ELLIPSE.

#### SIMPLE PROPERTIES OF THE ELLIPSE.

116. The **Ellipse** is the locus of a point, the sum of whose distances from two fixed points is constant.

The fixed points are called **Foci**; and the distance from any point of the curve to a focus is called a **Focal Radius**.

The constant sum is denoted by  $2a$ , and the distance between the foci by  $2c$ .

The fraction  $\frac{c}{a}$  is called the **Eccentricity**, and is represented by the letter  $e$ . Therefore  $c = ae$ .

In the ellipse  $a > c$ ; that is,  $e < 1$ .

If  $a = c$ , the locus is simply the limited straight line joining the foci.

If  $a < c$ , from the definition it is clear that there is no locus.

117. *To construct an ellipse, having given the foci and the constant sum  $2a$ .*

I. *By Motion.* Fix pins in the paper at the foci. Tie a string to them, making the length of the string exactly equal to  $2a$ . Then press a pencil against the string so as to make it tense, and move the pencil, keeping the string constantly stretched. The point of the pencil will trace the required ellipse; for in every position the sum of the distances from the point of the pencil to the foci is equal to the length of the string.

II. *By Points.* Let  $F, F'$  be the foci; then  $FF' = 2c$ .

Bisect  $FF'$  at  $O$ , and from  $O$  lay off  $OA = OA' = a$ .

Then  $AA' = 2a$ ,  $F'A' = FA$ ,

$$A'F + A'F' = A'F + FA = 2a,$$

$$AF' + AF = AF' + F'A' = 2a.$$

Therefore  $A$  and  $A'$  are points of the curve.

Between  $A$  and  $A'$  mark any point  $X$ ; then describe two arcs, one with  $F$  as centre and  $AX$  as radius, the other with  $F'$  as centre and  $A'X$  as radius: the intersections  $P, Q$  of

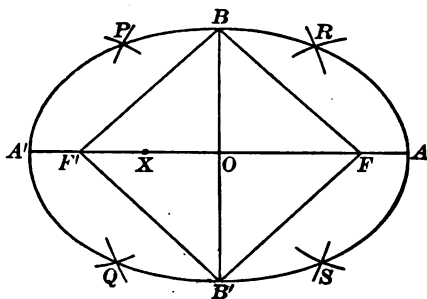


Fig. 54.

these arcs are points of the curve. By merely interchanging the radii, two more points,  $R, S$ , may be found.

After a sufficient number of points has been obtained, draw a continuous curve through them.

118. The line  $AA'$  is the **Transverse** or **Major Axis**,  $A, A'$  the **Vertices**, and  $O$  the **Centre** of the curve.

The line  $BB'$ , perpendicular to the major axis at  $O$ , is the **Conjugate** or **Minor Axis**; its length is denoted by  $2b$ .

Show that  $B$  and  $B'$  are equidistant from the foci, that  $BF = a$ , that  $BO = b$ , and that  $a^2 = b^2 + c^2 = b^2 + a^2e^2$ .

119. To find the equation of the ellipse, having given the foci and the constant sum  $2a$ .

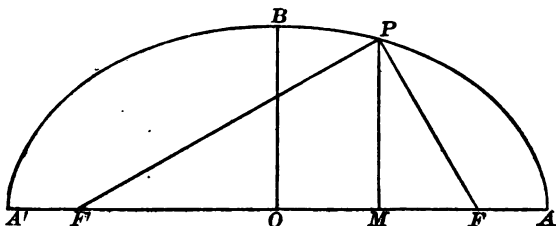


Fig. 55.

Take the line  $AA'$  (Fig. 55), passing through the foci, as the axis of  $x$ , and the point  $O$ , half way between the foci, as origin. Let  $P$  be any point  $(x, y)$  of the curve, and let  $r, r'$  denote the focal radii of  $P$ . Then from the definition of the curve, and from the right triangles  $F'PM, FPM$ ,

$$r'^2 = y^2 + (c + x)^2 \quad (1)$$

$$r^2 = y^2 + (c - x)^2 \quad (2)$$

$$\text{By addition,} \quad r'^2 + r^2 = 2(x^2 + y^2 + c^2). \quad (3)$$

$$\text{By subtraction,} \quad r'^2 - r^2 = 4cx. \quad (4)$$

$$\text{But} \quad r' + r = 2a. \quad (5)$$

$$\text{By division,} \quad r' - r = \frac{2cx}{a}. \quad (6)$$

$$\text{By subtraction,} \quad r = a - \frac{cx}{a} = [a - ex]. \quad (7)$$

$$\text{By addition,} \quad r' = a + \frac{cx}{a} = [a + ex]. \quad (8)$$

$$\text{Put in (3),} \quad 2\left(a^2 + \frac{c^2 x^2}{a^2}\right) = 2(x^2 + y^2 + c^2). \quad (9)$$

Reduce, and substitute  $b^2$  in place of  $a^2 - c^2$  (§ 118),

$$b^2 x^2 + a^2 y^2 = a^2 b^2,$$

$$\text{or} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \quad [30]$$

COR. If the transverse axis were on the axis of  $y$ , and the conjugate on the axis of  $x$ , the equation of the ellipse would evidently be

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1. \quad (10)$$

120. To trace the form of the curve from its equation.

The intercepts on the axis of  $x$  are  $+a$  and  $-a$ ; on the axis of  $y$ ,  $+b$  and  $-b$ .

Only the squares of the variables  $x$  and  $y$  appear in the equation; hence, if it is satisfied by a point  $(x, y)$ , it will also be satisfied by the points  $(x, -y)$ ,  $(-x, y)$ ,  $(-x, -y)$ . Therefore we infer that

- (i.) The curve is symmetrical with respect to the axis of  $x$ .
- (ii.) The curve is symmetrical with respect to the axis of  $y$ .
- (iii.) The curve is symmetrical with respect to the centre  $O$ , which bisects every chord passing through it. This explains why  $O$  is called the centre.

Since the sum of  $\left(\frac{x}{a}\right)^2$  and  $\left(\frac{y}{b}\right)^2$  is 1, neither of these squares can exceed 1; therefore the maximum value of  $x$  is  $+a$ , and the minimum value  $-a$ , while the corresponding values of  $y$  are  $+b$  and  $-b$ . Therefore the curve is wholly contained within the rectangle whose sides are  $x = \pm a$ ,  $y = \pm b$ .

121. To trace the changes in the form of the ellipse when the semi-axes are supposed to change.

Let  $a$  be regarded as a constant, and  $b$  as a variable.

(i.) Suppose  $b$  to increase. Then  $c$  decreases (since  $c^2 = a^2 - b^2$ ),  $e$  decreases, the foci approach the centre, and the ellipse approaches the circle.

(ii.) Let  $b = a$ . Then  $c = 0$ ,  $e = 0$ , the foci coincide with the centre, the ellipse becomes a circle of radius  $a$ , and equation [30] becomes the equation of the circle,

$$x^2 + y^2 = a^2.$$



(iii.) If we suppose  $b$  to decrease to 0 ( $a$  remaining constant),  $c$  will increase to  $a$ ,  $e$  will increase to 1, while the curve will approach, and finally coincide with, the major axis, its equation at the same time becoming  $y = 0$ .

122. Let  $(x_1, y_1)$  and  $(x_2, y_2)$  be any two points on the ellipse  $b^2x^2 + a^2y^2 = a^2b^2$ ; then we have

$$y_1^2 = \frac{b^2}{a^2}(a^2 - x_1^2), \quad y_2^2 = \frac{b^2}{a^2}(a^2 - x_2^2).$$

Dividing and factoring, we have

$$y_1^2 : y_2^2 :: (a - x_1)(a + x_1) : (a - x_2)(a + x_2).$$

That is, *the squares of any two ordinates of the ellipse are to each other as the products of the segments into which they divide the major axis.*

123. It follows from § 119 that a point  $(h, k)$  is on the ellipse represented by the equation [30], provided

$$\frac{h^2}{a^2} + \frac{k^2}{b^2} - 1 = 0.$$

It may be shown by reasoning similar to that employed in § 96 that the point  $(h, k)$  is *outside* or *inside* the curve, according as  $\frac{h^2}{a^2} + \frac{k^2}{b^2} - 1$  is *positive* or *negative*.

124. If  $A, B, C$  all have the same sign, every equation of the form

$$Ax^2 + By^2 = C \tag{1}$$

may be reduced to the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ or } \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1.$$

Hence every equation of the form of (1) represents an ellipse whose semi-axes are  $\sqrt{\frac{C}{A}}$  and  $\sqrt{\frac{C}{B}}$ . The transverse axis lies on the axis of  $x$  or the axis of  $y$ , according as  $A$  is *less* or *greater* than  $B$ .

**125.** The chord passing through either focus perpendicular to the major axis is called the **Latus Rectum** or **Parameter**.

To find its length, put  $x = c$  in the equation of the ellipse.

Then 
$$y^2 = \frac{b^2}{a^2}(a^2 - c^2) = \frac{b^4}{a^2}, \quad y = \pm \frac{b^2}{a}.$$

Therefore the latus rectum  $= \frac{2b^2}{a} = \left[ \frac{4b^2}{2a} \right].$

Forming a proportion from this equation, we have

$$2a : 2b :: 2b : \text{latus rectum};$$

that is, *the latus rectum is a third proportional to the major and the minor axis.*

**126.** The circle having for diameter the major axis of the ellipse is called the **Auxiliary Circle**; its equation is

$$x^2 + y^2 = a^2.$$

The circle having for diameter the minor axis is called the **Minor Auxiliary Circle**; its equation is

$$x^2 + y^2 = b^2.$$

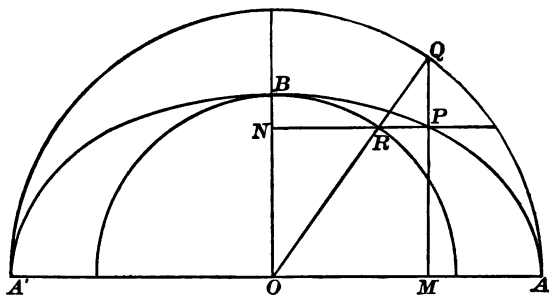


Fig. 56.

If  $P$  (Fig. 56) is any point of an ellipse, and the ordinate  $MP$  produced meets the auxiliary circle in  $Q$ , the point  $Q$  is said to *correspond* to the point  $P$ .

The angle  $QOM$  is called the **Eccentric Angle** of the point  $P$ , and denoted by the letter  $\phi$ .

127. Let  $y, y'$  represent the ordinates of points in an ellipse and the auxiliary circle respectively, corresponding to the same abscissa  $x$ . Then from the equations of the two curves we have

$$y = \frac{b}{a} \sqrt{a^2 - x^2}, \quad y' = \sqrt{a^2 - x^2}.$$

Whence

$$y : y' = b : a,$$

or, *the ordinates of the ellipse and the auxiliary circle, corresponding to a common abscissa, are to each other in the constant ratio of the semi-minor and semi-major axes of the ellipse.*

128. The principle of § 127 furnishes the following easy method of constructing an ellipse by points when its axes are given :

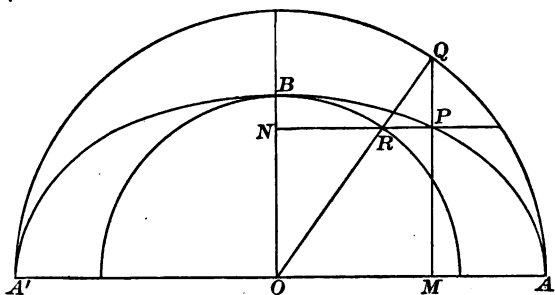


Fig. 57.

Construct both the major and minor auxiliary circles (Fig. 57); draw any radius cutting the circles in  $R$  and  $Q$ ; through  $Q$  draw a line parallel to  $OB$ , and through  $R$  draw a line parallel to  $OA$ ; the intersection  $P$  of these lines is a point on the ellipse. For we have

$$MP : MQ = OR : OQ,$$

or

$$MP : y' = b : a.$$

From this proportion and that in § 127, we have  $MP = y$ ; hence  $P$  is a point on the ellipse. In like manner any number of points may be found.

COR. From Fig. 57, we have

$$\left. \begin{aligned} x &= OM = OQ \cos \phi = a \cos \phi, \\ y &= MP = ON = OR \sin \phi = b \sin \phi. \end{aligned} \right\} \quad (1)$$

Equations (1), which express the co-ordinates of any point of the ellipse in terms of its eccentric angle, may be used as the equations of the ellipse. To obtain from them the common equation, we have

$$\frac{x}{a} = \cos \phi, \text{ and } \frac{y}{b} = \sin \phi.$$

$$\text{Therefore } \frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2 \phi + \sin^2 \phi = 1.$$

**129.** *To find the area of an ellipse.*

Divide the semi-major axis  $OA$  (Fig. 58) into any number of equal parts, through any two adjacent points of division  $M, N$  erect ordinates, and let the ordinate through  $M$  meet

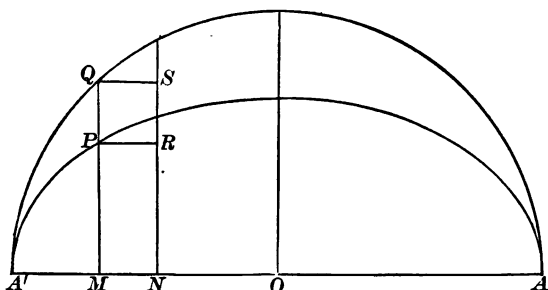


Fig. 58.

the ellipse in  $P$  and the auxiliary circle in  $Q$ . Through  $P, Q$  draw parallels to the axis of  $x$ , meeting the other ordinate in  $R, S$ , respectively. Then (§ 127)

$$\frac{\text{area of rectangle } MPRN}{\text{area of rectangle } MQSN} = \frac{MP}{MQ} = \frac{b}{a}.$$

And a similar proportion holds true for every corresponding pair of rectangles.

Therefore, by the Theory of Proportion,

$$\frac{\text{sum of rectangles in ellipse}}{\text{sum of rectangles in circle}} = \frac{b}{a}.$$

This relation holds true however great the number of rectangles. The greater their number, the nearer does the sum of their areas approach the area of the elliptic quadrant in one case, and the circular quadrant in the other. In other words, these two quadrants are the *limits* of the sums of the two series of rectangles: Therefore, by the fundamental theorem of limits,

$$\frac{\text{area of elliptic quadrant}}{\text{area of circular quadrant}} = \frac{b}{a}.$$

Multiplying both terms of the first ratio by 4,

$$\frac{\text{area of the ellipse}}{\text{area of the circle}} = \frac{b}{a}.$$

But the area of the circle  $= \pi a^2$ ; therefore

$$\text{area of the ellipse} = \pi ab. \quad [31]$$

### Ex. 32.

What are  $a$ ,  $b$ ,  $c$ , and  $e$  in the ellipse whose equation is

1.  $\frac{x^2}{25} + \frac{y^2}{16} = 1$  ?

2.  $x^2 + 2y^2 = 2$  ?

3.  $3x^2 + 4y^2 = 12$  ?

4.  $Ax^2 + By^2 = 1$  ?

5. Find the latus rectum of the ellipse  $3x^2 + 7y^2 = 18$ .

6. Find the eccentricity of an ellipse if its latus rectum is equal to one-half its minor axis.

What is the equation of an ellipse if

7. The axes are 12 and 8?
8. Major axis = 26, distance between foci = 24?
9. Sum of axes = 54, distance between foci = 18?
10. Latus rectum =  $\frac{32}{5}$ , eccentricity =  $\frac{3}{5}$ ?
11. Minor axis = 10, distance from focus to vertex = 1?
12. The curve passes through (1, 4) and (-6, 1)?
13. Major axis = 20, minor axis = distance between foci?
14. Sum of the focal radii of a point in the curve = 3 times the distance between the foci?
15. Prove that the semi-minor axis is a mean proportional between the segments of the major axis made by one of the foci.
16. What is the ratio of the two axes if the centre and foci divide the major axis into four equal parts?
17. For what point of an ellipse is the abscissa equal to the ordinate?

Find the intersections of the loci

18.  $3x^2 + 6y^2 = 11$  and  $y = x + 1$ .
19.  $2x^2 + 3y^2 = 14$  and  $y^2 = 4x$ .
20.  $x^2 + 7y^2 = 16$  and  $x^2 + y^2 = 10$ .
21. The ordinates of the circle  $x^2 + y^2 = r^2$  are bisected; find the locus of the points of bisection.
22. A straight line  $AB$  so moves that the points  $A$  and  $B$  always touch two fixed perpendicular straight lines. Show that any point  $P$  in  $AB$  describes an ellipse, and find its equation.

23. What is the locus of  $Ax^2 + By^2 = C$  when  $C$  is zero? When is this locus imaginary?

24. Prove that the abscissas of the ellipse  $b^2x^2 + a^2y^2 = a^2b^2$  are to the corresponding abscissas of the minor auxiliary circle,  $x^2 + y^2 = b^2$ , as  $a : b$ .

25. Construct an ellipse by the method of § 128.

26. Construct an ellipse, having given  $c$  and  $b$ .

27. Construct the axes of an ellipse, having given the foci and one point of the curve.

28. Construct the minor axis and foci, having given the major axis (in magnitude and position) and one point of the curve.

29. A square is inscribed in the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Find the equations of the sides and the area of the square.

#### TANGENTS AND NORMALS.

130. To find the equations of a tangent and a normal to an ellipse, having given the point of contact  $(x_1, y_1)$ .

Taking the equation of the ellipse,

$$b^2x^2 + a^2y^2 = a^2b^2,$$

and the equation of the straight line through  $(x_1, y_1)$  and  $(x_2, y_2)$ ,

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1},$$

and proceeding as in § 64, we obtain as the equation of a secant through  $(x_1, y_1)$  and  $(x_2, y_2)$

$$\frac{y - y_1}{x - x_1} = -\frac{b^2(x_1 + x_2)}{a^2(y_1 + y_2)}.$$

Now make  $x_2 = x_1$ ,  $y_2 = y_1$ ; then the chord becomes a tangent, and

$$\frac{y - y_1}{x - x_1} = -\frac{b^2(x_1 + x_2)}{a^2(y_1 + y_2)}$$

becomes

$$\frac{y - y_1}{x - x_1} = -\frac{b^2 x_1}{a^2 y_1},$$

which reduces to  $\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1$ . [32]

From the equation above it appears that the value of the slope of the tangent, in terms of the co-ordinates of the point of contact, is

$$-\frac{b^2 x_1}{a^2 y_1}.$$

The normal is perpendicular to the tangent, and passes through  $(x_1, y_1)$ ; therefore its equation is easily found (by the method of § 45) to be

$$y - y_1 = \frac{a^2 y_1}{b^2 x_1} (x - x_1). \quad [33]$$

131. *To find the subtangent and subnormal.*

Making  $y = 0$  in [32] and [33], and then solving the equations for  $x$ , we obtain:

$$\text{Intercept of tangent on axis of } x = \frac{a^2}{x_1},$$

$$\text{Intercept of normal on axis of } x = \frac{c^2}{a^2} x_1 = e^2 x_1.$$

Whence the values of the subtangent and the subnormal (defined as in § 63) are easily found to be as follows:

$$\text{Subtangent} = \frac{x_1^2 - a^2}{x_1}, \quad [34]$$

$$\text{Subnormal} = -\frac{b^2}{a^2} x_1. \quad [35]$$



**132.** *If tangents to ellipses having a common major axis are drawn at points having a common abscissa, they will meet on the axis of  $x$ .*

For in all these ellipses the values of  $a$  and  $x$  are constant, and therefore (by § 131) the tangents all cut the same intercept from the axis of  $x$ .

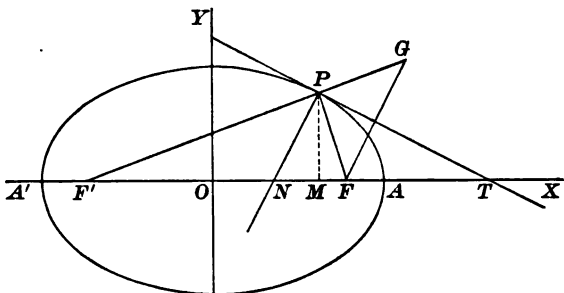


Fig. 59.

**133.** *The normal at any point of an ellipse bisects the angle formed by the focal radii.*

The values of the focal radii for the point  $P$  (Fig. 59) were found in § 119 to be

$$PF = a - ex_1, \quad PF' = a + ex_1.$$

If the normal through  $P$  meets the axis of  $x$  in  $N$ ,  $ON = e^2x$  (§ 131); and therefore

$$NF = c - e^2x_1 = ae - e^2x_1 = e(a - ex).$$

$$NF' = c + e^2x_1 = ae + e^2x_1 = e(a + ex).$$

Therefore  $NF : NF' = PF : PF'$ ,

or the normal divides the side  $FF'$  of the  $\triangle PFF'$  into two parts proportional to the other two sides. Therefore (by Geometry)  $FPN = F'PN$ .

The tangent  $PT$ , being perpendicular to the normal, must bisect the angle  $FPG$ , formed by one focal radius with the other produced.

**134.** *To draw a tangent and a normal through a given point of an ellipse.*

I. Let  $P$  (Fig. 60) be the given point. Describe the auxiliary circle, draw the ordinate  $PM$ , produce it to meet the circle in  $Q$ , draw  $QT$  tangent to the circle and meeting the axis of  $x$  in  $T$ , and join  $PT$ ; then  $PT$  is a tangent to the ellipse (§ 132). Draw  $PN \perp$  to  $PT$ ;  $PN$  is the normal at  $P$ .

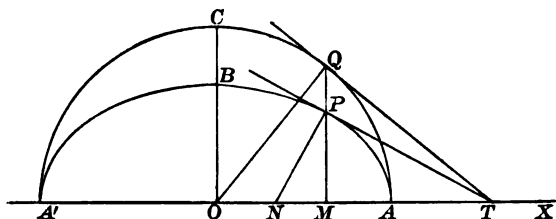


Fig. 60.

II. Draw the focal radii, and bisect the angles between them. The bisectors are the tangent and the normal at the point  $P$  (§ 133).

**135.** *To find the equation of a tangent to an ellipse in terms of its slope.*

This problem may be solved by finding under what condition the straight line

$$y = mx + c \quad (1)$$

will touch the ellipse  $b^2x^2 + a^2y^2 = a^2b^2$ . (2)

Eliminating  $y$  from (1) and (2), and then solving for  $x$ , we find two values of  $x$ :

$$x = \frac{-ma^2c \pm ab\sqrt{m^2a^2 + b^2 - c^2}}{m^2a^2 + b^2}.$$

These values will be equal if

$$m^2a^2 + b^2 - c^2 = 0, \text{ or } c = \pm \sqrt{m^2a^2 + b^2}.$$

If the two values of  $x$  are equal, the two values of  $y$  must also be equal from equation (1).

Therefore the two points in which the ellipse is cut by the line will coincide if  $c = \pm \sqrt{m^2 a^2 + b^2}$ .

Hence the straight line

$$y = mx \pm \sqrt{m^2 a^2 + b^2} \quad [36]$$

will touch the ellipse for all values of  $m$ .

Since either sign may be given to the radical, it follows that two tangents may be drawn to an ellipse, having the same slope.

**136.** *To find the locus of the intersection of two tangents to an ellipse which are perpendicular to each other.*

Let the equations of the tangents be

$$y = mx + \sqrt{m^2 a^2 + b^2}, \quad (1)$$

$$y = m'x + \sqrt{m'^2 a^2 + b^2}. \quad (2)$$

The condition to be satisfied is

$$mm' = -1, \text{ or } m' = -\frac{1}{m}.$$

If we substitute for  $m'$  in equation (2) its value in terms of  $m$ , the equations of the tangents may be written

$$y - mx = \sqrt{m^2 a^2 + b^2}, \quad (3)$$

$$my + x = \sqrt{a^2 + m^2 b^2}. \quad (4)$$

The co-ordinates,  $x$  and  $y$ , of the intersection of the tangents satisfy both (3) and (4); but before we can find the constant relation between them we must first eliminate the variable  $m$ .

This is most easily done by adding the squares of the two equations; the result is

$$(1 + m^2)x^2 + (1 + m^2)y^2 = (1 + m^2)(a^2 + b^2),$$

or 
$$x^2 + y^2 = a^2 + b^2.$$

The required locus is therefore a circle. This circle is called the **Director Circle** of the ellipse.

## Ex. 33.

1. What are the equations of the tangent and the normal to the ellipse  $2x^2 + 3y^2 = 35$  at the points whose abscissa  $= 2$ ?

2. What are the equations of the tangent and the normal to the ellipse  $4x^2 + 9y^2 = 36$  at the points whose abscissa  $= -\frac{3}{2}$ ?

3. Find the equations of the tangent and the normal to the ellipse  $x^2 + 4y^2 = 20$  at the point of contact  $(2, 2)$ . Also find the subtangent and the subnormal.

4. Show that the line  $y = x + \sqrt{\frac{5}{3}}$  touches the ellipse  $2x^2 + 3y^2 = 1$ .

5. Required the condition which must be satisfied in order that the straight line  $\frac{x}{m} + \frac{y}{n} = 1$  may touch the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ .

6. In an ellipse the subtangent for the point  $(3, \frac{1}{3})$  is  $-\frac{1}{8}$ , the eccentricity  $= \frac{4}{5}$ . What is the equation of the ellipse?

7. What is the equation of a tangent to the ellipse  $9x^2 + 64y^2 = 576$  parallel to the line  $2y = x$ ?

8. Find the equation of a tangent to the ellipse  $3x^2 + 5y^2 = 15$  parallel to the line  $4x - 3y - 1 = 0$ .

9. In what points do the tangents which are equally inclined to the axes touch the ellipse  $b^2x^2 + a^2y^2 = a^2b^2$ ?

10. Through what point of the ellipse  $b^2x^2 + a^2y^2 = a^2b^2$  must a tangent and a normal be drawn in order that they may form, with the axis of  $x$  as base, an isosceles triangle?

11. Through a point of the ellipse  $b^2x^2 + a^2y^2 = a^2b^2$ , and the corresponding point of the auxiliary circle  $x^2 + y^2 = a^2$ , normals are drawn. What is the ratio of the subnormals?

12. For what points of the ellipse  $b^2x^2 + a^2y^2 = a^2b^2$  is the subtangent equal numerically to the abscissa of the point of contact?

13. Find the equations of tangents drawn from the point (3, 4) to the ellipse  $16x^2 + 25y^2 = 400$ .

14. What are the equations of the tangents drawn through the extremities of the latera recta of the ellipse  $4x^2 + 9y^2 = 36a^2$ ?

15. What is the distance from the centre of an ellipse to a tangent making the angle  $\phi$  with the major axis?

16. What is the area of the triangle formed by the tangent in the last problem and the axes of co-ordinates?

17. From the point where the auxiliary circle cuts the minor axis produced tangents are drawn to the ellipse. Find the points of contact.

18. Prove that the tangents drawn through the ends of a chord through the centre are parallel.

19. Find the locus of the foot of a perpendicular dropped from the focus to a tangent.

#### Ex. 34. (Review.)

1. Given the ellipse  $36x^2 + 100y^2 = 3600$ . Find the equations and the lengths of focal radii drawn to the point  $(8, \frac{16}{5})$ .

2. Is the point (2, 1) within or without the ellipse  $2x^2 + 3y^2 = 12$ ?

Find the eccentricity of an ellipse

3. If the equation is  $2x^2 + 3y^2 = 12$ .

4. If the angle  $FBF' = 90^\circ$  (see Fig. 54).

5. Show that  $4x^2 - 8x + 9y^2 - 36y + 4 = 0$ , or  $4(x-1)^2 + 9(y-2)^2 = 36$ , is an ellipse whose centre is (1, 2), and semi-axes 3 and 2.

Find the equations of tangents to an ellipse

6. If they make equal intercepts on the axes.

7. If they are parallel to  $BF$  (Fig. 54).

8. Which are parallel to the line  $\frac{x}{a} + \frac{y}{b} = 1$  ( $a$  and  $b$  being the semi-axes).

9. Find the equation of a tangent in terms of the eccentric angle  $\phi$  of the point of contact.

Find the distance from the centre of an ellipse to

10. A tangent through the point of contact  $(x_1, y_1)$ .

11. A tangent making the angle  $\phi$  with the axis of  $x$ .

12. In what ratio is the abscissa of a point divided by the normal at that point?

13. At the point  $(x_1, y_1)$  of an ellipse a normal is drawn. What is the product of the segments into which it divides the major axis?

14. Find the length of  $PN$  (Fig. 59).

15. Determine the value of the eccentric angle at the end of the latus rectum.

Prove that the semi-minor axis  $b$  of an ellipse is a mean proportional between

16. The distances from the foci to a tangent.

17. A normal and the distance from the centre to the corresponding tangent.

Determine and describe the loci of the following points:

18. The middle point of that portion of a tangent contained between the tangents drawn through the vertices.

19. The middle point of a perpendicular dropped from a point of the circle  $(x - a)^2 + y^2 = r^2$  to the axis of  $y$ .

20. The middle point of a chord of the ellipse  $b^2x^2 + a^2y^2 = a^2b^2$  drawn through the positive end of the minor axis.

21. The vertex of a triangle whose base  $2c$  and sum of the other sides  $2s$  are given.

22. The vertex of a triangle, having given the base  $2c$  and the product  $k$  of the tangents of the angles at the base.

23. The symmetrical point of the right-hand focus of an ellipse with respect to a tangent.

### SUPPLEMENTARY PROPOSITIONS.

137. *Two distinct, two coincident, or no tangents can be drawn to an ellipse through any point  $(h, k)$ , according as the point is without, on, or within the curve.*

Let the tangent  $y = mx + \sqrt{m^2a^2 + b^2}$  pass through the point  $(h, k)$ ; then

$$k = mh + \sqrt{m^2a^2 + b^2},$$

or

$$(h^2 - a^2)m^2 - 2hkm + k^2 - b^2 = 0.$$

$$\therefore m = \frac{hk \pm \sqrt{b^2h^2 + a^2k^2 - a^2b^2}}{h^2 - a^2}. \quad (1)$$

Hence there will be two distinct, two coincident, or no tangents through  $(h, k)$ , according as  $b^2h^2 + a^2k^2 - a^2b^2 >, =$ , or  $< 0$ ; that is, according as  $(h, k)$  is without, on, or within the ellipse.

138. *To find the equation of the chord of contact of the two tangents drawn from an external point  $(h, k)$  to the ellipse,*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

By a course of reasoning similar to that employed in §§ 71 and 104, let the student prove that the required equation is

$$\frac{hx}{a^2} + \frac{ky}{b^2} = 1.$$

139. To find the equation of the polar of the pole  $(h, k)$ , with regard to the ellipse.

By a course of reasoning similar to that employed in §§ 72 and 105, let the student prove that the required equation is:

$$\frac{hx}{a^2} + \frac{ky}{b^2} = 1.$$

COR. The tangent and chord of contact are particular cases of the polar. Proposition 74 holds true of poles and polars with regard to the ellipse.

140. To draw a tangent to an ellipse from a given point  $P$  outside the curve.

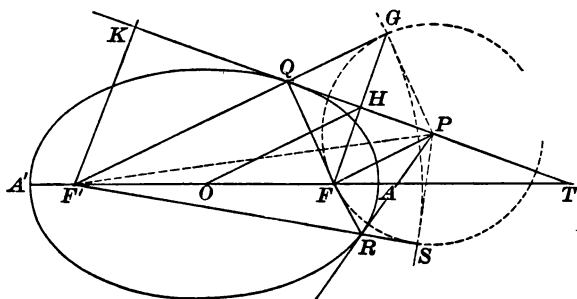


Fig. 61.

Suppose the problem solved, and let the tangent meet the ellipse at  $Q$  (Fig. 61). If  $F'Q$  be produced to  $G$ , making  $QG = QF$ , then  $\triangle FQG$  is isosceles; now  $\angle FQP = \angle GQP$  (§ 133); therefore  $PQ$  is perpendicular to  $FG$  at its middle point; therefore  $P$  is equidistant from  $F$  and  $G$ . This reduces the problem to determining the point  $G$ .

Since  $F'G = 2a$ ,  $G$  lies in the circle with  $F'$  as centre and  $2a$  as radius. And  $G$  also lies in the circle with  $P$  as centre and  $PF$  as radius. Hence the construction is obvious.



**141.** *To find the locus of the middle points of any system of parallel chords in the ellipse.*

Let any one of the parallel chords  $y = mx + c$  meet the ellipse

$$b^2x^2 + a^2y^2 = a^2b^2$$

in the points  $(x_1, y_1)$  and  $(x_2, y_2)$ ; then by (3) of § 130,

$$m = -\frac{b^2(x_1 + x_2)}{a^2(y_1 + y_2)}. \quad (1)$$

If  $(x, y)$  is the middle point,  $2x = x_1 + x_2$ ,  $2y = y_1 + y_2$ , and (1) becomes

$$m = -\frac{b^2x}{a^2y},$$

or

$$y = -\frac{b^2x}{a^2m}. \quad (2)$$

This relation holds true for the middle points of all the chords; therefore it is the equation required.

From (2), we see that any straight line passing through the centre of an ellipse is a *diameter*.

**142.** Let  $m'$  denote the slope of the diameter of the chords whose slope is  $m$ ; then from (2) of § 141

$$m' = -\frac{b^2}{ma^2}, \text{ or } mm' = -\frac{b^2}{a^2}. \quad [37]$$

Thus [37] is the equation of condition that the diameter  $y = m'x$  bisects all chords parallel to the diameter  $y = mx$ ; but [37] is evidently also the equation of condition that  $y = mx$  bisects all chords parallel to  $y = m'x$ ; hence,

*If one diameter bisects all chords parallel to another, the second diameter bisects all chords parallel to the first.*

Two such diameters are called **Conjugate Diameters**.

**COR.** From [37] the slopes of two conjugate diameters must have opposite signs; hence, *two conjugate diameters of an ellipse lie on opposite sides of the minor axis.*

143. Let a straight line cutting the ellipse in  $P$  and  $Q$  move parallel to itself till  $P$  and  $Q$  coincide with the end of the diameter bisecting  $PQ$ ; then the straight line becomes the tangent at the end of the diameter. Therefore

*The tangents at the extremities of any diameter are parallel to the chords of that diameter, and also to its conjugate diameter.*

144. Let  $POP'$  and  $ROR'$  (Fig. 62) be two conjugate diameters meeting the ellipse in the points  $P(x_1, y_1)$  and  $R(x_2, y_2)$ . The slope of the tangent through  $P$  is  $-\frac{b^2x_1}{a^2y_1}$ ; hence the equation of the diameter  $ROR'$ , which is parallel to this tangent (§ 143), is

$$y = -\frac{b^2x_1}{a^2y_1}x, \text{ or } \frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 0. \quad (1)$$

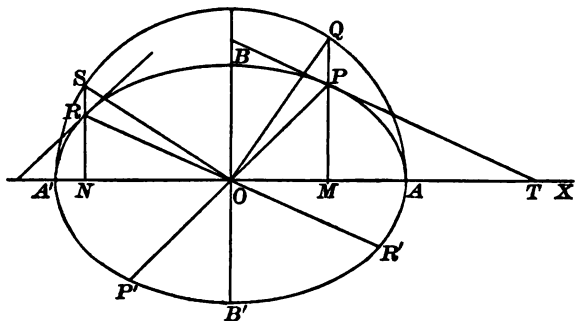


Fig. 62.

Now  $R(x_2, y_2)$  is on (1), and also on the ellipse; hence we have

$$\frac{x_1x_2}{a^2} + \frac{y_1y_2}{b^2} = 0, \quad (2)$$

and 
$$\frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} = 1. \quad (3)$$

Solving (2) and (3) for  $x_2$  and  $y_2$ , we obtain

$$x_2 = \mp \frac{a}{b}y_1, \quad y_2 = \pm \frac{b}{a}x_1. \quad (4)$$

The upper signs give the co-ordinates of  $R$ , and the lower those of  $R'$  in terms of  $x_1$  and  $y_1$ .

Equation (2) is the condition which must be satisfied by the co-ordinates of the extremities of every pair of conjugate diameters.

145. Denoting the semi-conjugate diameters  $OP$  and  $OR$  (Fig. 62) by  $a'$  and  $b'$  respectively, we have

$$\begin{aligned} a'^2 &= x_1^2 + y_1^2 = x_1^2 + \frac{b^2}{a^2}(a^2 - x_1^2) \\ &= b^2 + \frac{a^2 - b^2}{a^2}x_1^2 = b^2 + e^2x_1^2, \end{aligned} \quad (1)$$

$$\begin{aligned} \text{and} \quad b'^2 &= x_2^2 + y_2^2 = \frac{a^2}{b^2}y_1^2 + \frac{b^2}{a^2}x_1^2 \quad \S 144 \\ &= a^2 - x_1^2 + \frac{b^2}{a^2}x_1^2 = a^2 - e^2x_1^2. \end{aligned} \quad (2)$$

Adding (1) and (2), we have

$$a'^2 + b'^2 = a^2 + b^2.$$

That is, *the sum of the squares of any pair of semi-conjugate diameters is equal to the sum of the squares of the semi-axis.*

Equations (1) and (2) express the lengths of the semi-conjugate diameters  $a'$  and  $b'$  in terms of  $a$ ,  $b$ , and  $x_1$  (the abscissa of the extremity of  $a'$ ).

146. Let the ordinates of the extremities  $P$ ,  $R$  (Fig. 62) of two conjugate diameters meet the auxiliary circle in  $Q$ ,  $S$  respectively, join  $QO$  and  $SO$ , and denote  $\angle QOX$  by  $\phi$ ,  $\angle SOX$  by  $\phi'$ . Then the values of the co-ordinates of  $P$  and  $R$  are (§ 127).

$$\begin{aligned} x_1 &= a \cos \phi, & x_2 &= a \cos \phi', \\ y_1 &= b \sin \phi, & y_2 &= b \sin \phi'. \end{aligned}$$

Whence, by substitution in equation (2) of § 144, we obtain

$$\cos \phi \cos \phi' + \sin \phi \sin \phi' = 0.$$

Therefore  $\cos(\phi' - \phi) = 0$ , or  $\phi' - \phi = \frac{1}{2}\pi$ .

That is, the difference of the eccentric angles corresponding to the ends of two conjugate diameters is equal to a right angle.

COR. The angle  $POR$  (Fig. 62) is obtuse, since  $QOS = \frac{1}{2}\pi$ .

147. To find the angle formed by two conjugate semi-diameters, whose lengths  $a'$ ,  $b'$  are given.

Let the semi-diameters make the angles  $\alpha$ ,  $\beta$  respectively with the axis of  $x$ , and let  $\theta$  denote the required angle. Then if  $(x_1, y_1)$  and  $(x_2, y_2)$  are the extremities of  $a'$  and  $b'$  respectively,

$$\begin{aligned}\sin \alpha &= \frac{y_1}{a'}, & \sin \beta &= \frac{y_2}{b'} = \frac{bx_1}{x'b'}, \\ \cos \alpha &= \frac{x_1}{a'}, & \cos \beta &= \frac{x_2}{b'} = -\frac{ay_1}{bb'}, \\ \sin \theta &= \sin(\beta - \alpha) \\ &= \sin \beta \cos \alpha - \cos \beta \sin \alpha \\ &= \frac{b^2 x_1^2 + a^2 y_1^2}{aba'b'} = \frac{a^2 b^2}{aba'b'} \\ &= \frac{ab}{a'b'}.\end{aligned}\tag{1}$$

COR. 1. Clearing (1) of fractions, we have

$$a'b' \sin \theta = ab,$$

which shows that the area of the parallelogram  $HEKR$  is equal to the rectangle  $LMQN$ . ( $\square CDRS = a'b' \sin \theta$ )

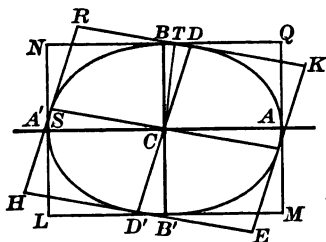


Fig. 63.

That is, the parallelogram formed by tangents at the extremities of any pair of conjugate diameters is equal to the rectangle on the axes.

COR. 2. If  $CT$  (Fig. 63) be perpendicular to the tangent  $RR$ ; then

$$CT = CD \sin CDR = a' \sin \theta = \frac{ab}{b'}.$$

148. The lines joining any point of an ellipse to the ends of any diameter are called **Supplemental Chords**.

Let  $PQ$ ,  $P'Q$  be two supplemental chords (Fig. 64). Through the centre  $O$  draw  $OR$  parallel to  $P'Q$ , and meeting  $PQ$  in  $R$ ; also  $OR'$  parallel to  $PQ$ , and meeting  $P'Q$  in  $R'$ .

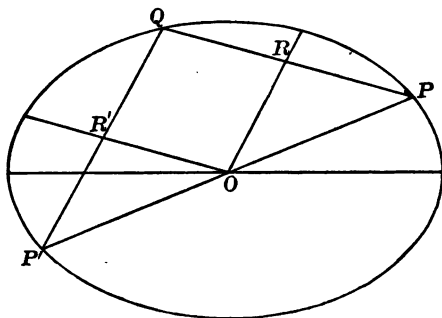


Fig. 64.

Since  $O$  is the middle point of  $PP'$ , and  $OR$  is drawn parallel to  $P'Q$ , and  $OR'$  is drawn parallel to  $PQ$ ,  $R$  and  $R'$  are the middle points of  $QP$ ,  $QP'$  respectively. Therefore  $OR$  will bisect all chords parallel to  $QP$ , and  $OR'$  will bisect all chords parallel to  $QP'$ . Hence  $OR$ ,  $OR'$  are conjugate diameters.

Therefore *the diameters parallel to a pair of supplemental chords are conjugate diameters.*

COR. 1. This principle affords the following easy method of drawing a pair of conjugate diameters which shall include a given angle.

On the transverse axis  $AA'$  describe a segment of a circle which shall include the given angle. Let the arc of this

segment cut the ellipse in  $Q$  and  $S$ ; then the diameters parallel to  $QA$  and  $QA'$ , or  $SA$  and  $SA'$  are conjugate and include the required angle.

COR. 2. If  $B$  be the upper vertex of the conjugate axis, the conjugate diameters parallel to  $BA$  and  $BA'$  will evidently be equal, and will lie on the diagonals of the rectangle on the axes of the ellipse.

149. *To find the equation of an ellipse referred to a pair of conjugate diameters as axes.*

Since each of two conjugate diameters of the ellipse bisects the chords parallel to the other, the curve is (obliquely) symmetrical with respect to each of the new axes; hence, as the required equation is of the second degree, it contains only the squares of  $x$  and  $y$ , and is of the form

$$Ax^2 + By^2 = C. \quad (1)$$

The intercepts of the curve on the new axes are the semi-conjugate diameters. Denoting them by  $a'$  and  $b'$ , we have

$$A = \frac{C}{a'^2}, \quad B = \frac{C}{b'^2}.$$

Substituting these values in (1), we obtain

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1, \quad [38]$$

which is the required equation in terms of the semi-conjugate diameters.

This equation has the same form as the equation referred to the axes of the curve; whence it follows that formulas derived from equation [30], by processes which do not presuppose the axes of co-ordinates to be rectangular, hold true when we employ as axes two conjugate diameters.

For example, the equation of a tangent at the point  $(x_1, y_1)$ , referred to the semi-conjugate diameters  $a'$  and  $b'$ , is

$$\frac{x_1 x}{a'^2} + \frac{y_1 y}{b'^2} = 1.$$

150. To construct the polar of a focus.

Since the polar of  $(h, k)$  is  $\frac{hx}{a^2} + \frac{ky}{b^2} = 1$ , the polar of the focus  $(ae, 0)$  is

$$aex = a^2, \text{ or } x = \frac{a}{e}.$$

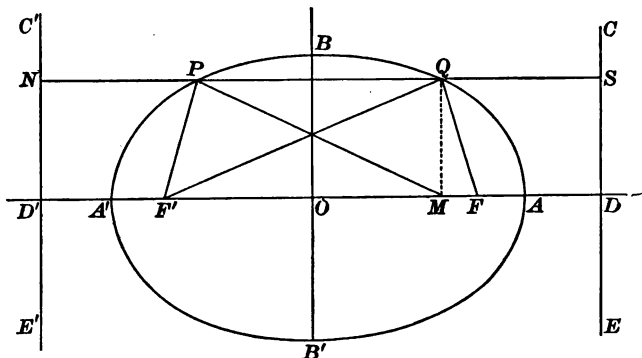


Fig. 65.

Hence  $ae : a = a : x$ .

Therefore, if  $OD$  be taken so that

$$OF : OA = OA : OD,$$

and  $DC$  be drawn perpendicular to  $OD$ ,  $DC$  will be the polar of the focus  $F$ .

The polar of a focus is called a **Directrix** of the ellipse. Hence  $DC$  is the directrix corresponding to the focus  $F$ .

In like manner we may construct  $E'C'$ , or the *directrix* corresponding to the focus  $F'$ .

COR. Let  $Q(x, y)$  be any point on the ellipse; then

$$QS = OD - OM = \frac{a}{e} - x = \frac{a - ex}{e} = \frac{FQ}{e}.$$

Hence  $e = FQ \div QS$ .

That is, *the distances of any point on the ellipse from a focus, and the corresponding directrix, bear the constant ratio  $e$ .*

Whence the ellipse is often defined as :

*The locus of a point which moves so that its distances from a fixed point and a fixed straight line bear a constant ratio less than unity.*

151. To find the polar equation of the ellipse, the left-hand focus being taken as the pole.

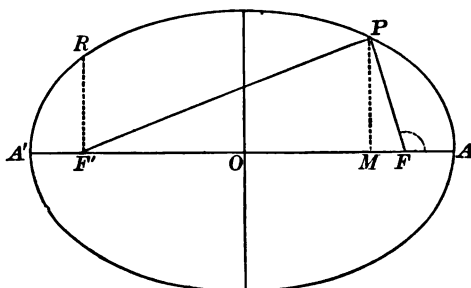


Fig. 66.

Let  $P$  be any point  $(\rho, \theta)$  of the ellipse ; then, from equation (8) of § 119, we have

$$\rho = a + ex. \quad (1)$$

Now  $x = OM = F'M - F'O = \rho \cos \theta - ae$ .

Substituting this value of  $x$  in (1), we have

$$\rho = a + e\rho \cos \theta - ae^2.$$

$$\text{Whence } \rho = \frac{a(1-e^2)}{1-e \cos \theta}. \quad [39]$$

COR. Since  $e < 1$ , and  $\cos \theta$  cannot exceed unity,  $\rho$  is always positive.

$$\text{For } \theta = 0, \quad \rho = \frac{a(1-e^2)}{1-e} = a + ae = F'A.$$

$$\text{For } \theta = \frac{1}{2}\pi, \quad \rho = a(1-e^2) = F'R = \text{semi-latus rectum.}$$



For  $\theta = \pi$ ,  $\rho = \frac{a(1-e^2)}{1+e} = a - ae = F'A'.$

For  $\theta = \frac{3}{2}\pi$ ,  $\rho = a(1-e^2) = \text{semi-latus rectum}.$

For  $\theta = 2\pi$ ,  $\rho = a + ae = F'A.$

While  $\theta$  increases from zero to  $\pi$ ,  $\rho$  decreases from  $a + ae$  to  $a - ae$ ; and while  $\theta$  increases from  $\pi$  to  $2\pi$ ,  $\rho$  increases from  $a - ae$  to  $a + ae$ .

If  $F$  be taken as the pole, the polar equation is

$$\rho = \frac{a(1-e^2)}{1+e \cos \theta}.$$

### Ex. 35.

1. Find the area of the ellipse  $x^2 + 4y^2 = 16$ .
2. Find the distances of the directrices from the centre, in Ex. 1.
3. What is the equation of the polar of the point (5, 7) with respect to the ellipse  $4x^2 + 9y^2 = 36$ ?
4. Prove that a focal chord is perpendicular to the line which joins its pole to the focus. In what line does the pole lie?
5. Find the pole of the line  $Ax + By + C = 0$  with respect to the ellipse  $b^2x^2 + a^2y^2 = a^2b^2$ .
6. Each of the two tangents which can be drawn to an ellipse from any point on its directrix subtends a right angle at the focus.
7. The two tangents which can be drawn to an ellipse from any external point subtend equal angles at the focus.
8. Find the slope  $m_1$  of a diameter if the square of the diameter is (i.) an arithmetic, (ii.) a geometric, (iii.) an harmonic mean between the squares of the axes.
9. Given the length  $2l$  of a diameter, its inclination  $\theta$  to the axis, and the eccentricity; find the major and minor axes.

10. Tangents at the extremities of any chord intersect on the diameter which bisects that chord.

11. Tangents are drawn from  $(3, 2)$  to the ellipse  $x^2 + 4y^2 = 4$ . Find the equation of the chord of contact, and of the line which joins  $(3, 2)$  to the middle point of the chord.

12. Find the area of the rectangle whose sides are the two segments into which a focal chord is divided by the focus.

13. What is the equation of a chord in the ellipse  $13x^2 + 11y^2 = 143$  which passes through  $(1, 2)$  and is bisected by the diameter  $3x - 2y = 0$ ?

14. In the ellipse  $9x^2 + 36y^2 = 324$  find the equation of a chord passing through  $(4, 2)$  and bisected at this point.

15. Write the equations of diameters conjugate to the following lines :

$$x - y = 0, \quad x + y = 0, \quad ax = by, \quad ay = bx.$$

16. Show that the lines  $2x - y = 0$ ,  $x + 3y = 0$  are conjugate diameters in the ellipse  $2x^2 + 3y^2 = 4$ .

17. Find the equation of a diameter parallel to the normal at the point  $(x_1, y_1)$ , the semi-axes being  $a$  and  $b$ .

18. The rectangle of the focal perpendiculars upon any tangent is constant and equal to the square of the semi-minor axes.

19. The diagonals of the parallelogram in Fig. 63, § 147, are also conjugate diameters.

20. The angle between two semi-conjugate diameters is a maximum, when they are equal.

21. The eccentric angles corresponding to equal semi-conjugate diameters are  $45^\circ$  and  $135^\circ$ .

22. The polar of a point in a diameter is parallel to the conjugate diameter.

23. Find the equations of equal conjugate diameters.
24. The length of a semi-diameter is  $l$ ; find the equation of the conjugate diameter.
25. The angle between two equal conjugate diameters =  $120^\circ$ ; find the eccentricity of the ellipse.
26. Given a diameter, to construct the conjugate diameter.
27. To draw a tangent to a given ellipse parallel to a given straight line.
28. Given an ellipse; to find by construction the centre, foci, and axes.
29. Find the rectangular equation of the ellipse, taking the origin at the right-hand vertex.
30. Find the polar equation of an ellipse, taking as pole the left-hand focus.
31. Find the polar equation of the ellipse, taking the centre as pole.
32. If the centre of an ellipse is the point  $(4, 7)$ , and the major and minor axes are 14 and 8, find its equation, the axes being supposed parallel to the axes of co-ordinates.
33. The equation of an ellipse, the origin being at the left-hand vertex, is  $25x^2 + 81y^2 = 450x$ ; find the axes.
34. If the minor axis = 12, and the latus rectum = 5, what is the equation of the ellipse, the origin being taken at the left-hand vertex?
35. Find the eccentric angle  $\phi$  corresponding to the diameter whose length is  $2c$ .
36. At the intersection of the ellipse  $b^2x^2 + a^2y^2 = a^2b^2$  and the circle  $x^2 + y^2 = ab$  tangents are drawn to both curves. Find the angle between them.

37. How would you draw a normal to an ellipse from any point in the minor axis?

38. Find the equation of a chord bisected at a point  $(h, k)$ .

39. Prove that the length of a line drawn from the centre to a tangent, and parallel to either focal radius of the point of contact, is equal to the semi-major axis.

40. A circle described on a focal radius will touch the auxiliary circle.

41. Find the locus of the intersection of tangents drawn through the ends of conjugate diameters of an ellipse.

42. Find the locus of the middle point of the chord joining the ends of two conjugate diameters.

43. Find the locus of the vertex of a triangle whose base is the line joining the foci, and whose other sides are parallel to two conjugate diameters.

44. Show that  $4x^2 + y^2 + 8x - 2y + 1 = 0$  represents an ellipse; find its centre and axes.

45. If  $A$  and  $B$  have like signs, show that the locus of  $Ax^2 + By^2 + Dx + Ey + F = 0$  is in general an ellipse whose axes are parallel to the co-ordinate axes; and determine its semi-axes.

46. Find the locus of the centre of a circle which passes through the point  $(0, 3)$  and touches internally the circle  $x^2 + y^2 = 25$ .

## CHAPTER VII.

### THE HYPERBOLA.

#### SIMPLE PROPERTIES OF THE HYPERBOLA.

**152.** The **Hyperbola** is the locus of a point the difference of whose distances from two fixed points is constant.

The fixed points are called the **Foci**, and a line joining any point of the curve to a focus is called a **Focal Radius**.

The constant difference is denoted by  $2a$ , and the distance between the foci by  $2c$ .

The fraction  $\frac{c}{a}$  is called the **Eccentricity**, and is denoted by the letter  $e$ . Therefore  $c = ae$ .

Since the difference of two sides of a triangle is always less than the third side, we must have in the hyperbola

$$2a < 2c, \text{ or } a < c, \text{ or } e > 1.$$

**153.** *To construct an hyperbola, having given the foci, and the constant difference  $2a$ .*

I. *By Motion* (Fig. 67). Fasten one end of a ruler to one focus  $F'$  so that it can turn freely about  $F'$ . To the other end fasten a string. Make the length of the string less than that of the ruler by  $2a$ , and fasten the free end to the focus  $F$ . Press the string against the ruler by a pencil point  $P$ , and turn the ruler about  $F'$ .

The point  $P$  will describe one branch of an hyperbola. The other branch may be described in the same way by interchanging the fixed ends of the ruler and the string.

II. *By Points* (Fig. 68). Let  $F, F'$  be the foci; then

$$FF' = 2c.$$

Bisect  $FF'$  at  $O$ , and from  $O$  lay off  $OA = OA' = a$ .

Then  $AA' = 2a, FA = F'A'.$

$$AF' - AF = A'F' - A'F' = AA' = 2a.$$

$$A'F - A'F' = A'F - AF = AA' = 2a.$$

Therefore  $A$  and  $A'$  are points of the curve.

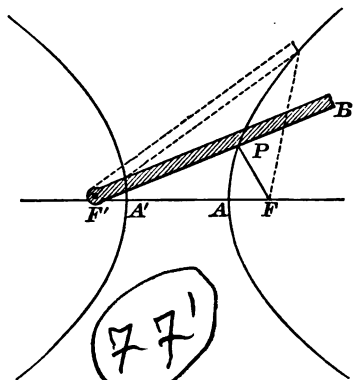


Fig. 67.

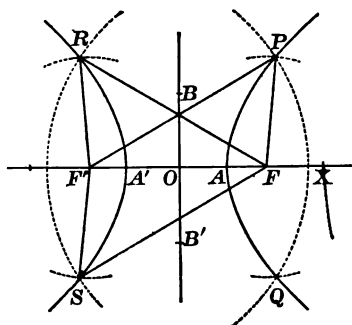


Fig. 68.

In  $AA'$  produced mark any point  $D$ ; then describe two arcs, the first with  $F$  as centre and  $AD$  as radius, the second with  $F'$  as centre and  $A'D$  as radius; the intersections  $P, Q$  of these arcs are points of the curve. By merely interchanging the radii, two more points  $R, S$  may be found.

Proceed in this way till a sufficient number of points has been obtained; then draw a smooth curve through them.

Through  $O$  draw  $BB' \perp$  to  $FF'$ ; since the difference of the distances of every point in the line  $BB'$  from the foci is 0, therefore the curve cannot cut the line  $BB'$ .

The locus evidently consists of two entirely distinct parts or *branches*, symmetrically placed with respect to the line  $BB'$ .

**154.** The point  $O$ , half way between the foci, is the **Centre**.

The line  $AA'$  passing through the foci and limited by the curve is the **Transverse Axis**.

The points  $A, A'$ , where the transverse axis meets the curve, are called the **Vertices**.

The transverse axis is equal to the constant difference  $2a$ , and is bisected by the centre (§ 153).

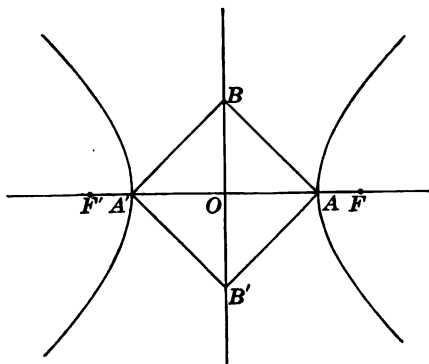


Fig. 69.

The line  $BB'$  passing through  $O$  perpendicular to  $AA'$  does not meet the curve (§ 153); but if  $B, B'$  are two points whose distances from the two vertices  $A, A'$  are each equal to  $c$ , then  $BB'$  is called the **Conjugate Axis**, and is denoted by  $2b$ .

Since  $\triangle AOB = \triangle A'O B'$ ,  $OB = OB' = b$ ; that is, the conjugate axis is bisected by the centre.

In the triangle  $AOB$ ,  $OA = a$ ,  $OB = b$ ,  $AB = c$ ; hence

$$c^2 = a^2 + b^2.$$

The chord passing through either focus perpendicular to the transverse axis is called the **Latus Rectum**, or **Parameter**.

**NOTE.** Since  $a$  and  $b$  are equal to the *legs* of a right triangle,  $a$  may be either greater or less than  $b$ ; hence the terms "*major*" and "*minor*" are not appropriate in the hyperbola.

155. By proceeding as in the case of the ellipse (§ 119), using  $r' - r = \pm 2a$  instead of  $r' + r = 2a$ , and substituting  $b^2$  for  $c^2 - a^2$ , we obtain as the equation of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \quad [40]$$

Thus the equations of the ellipse and hyperbola differ only in the sign of  $b^2$ ; that of the ellipse is changed into that of the hyperbola by substituting  $-b^2$  for  $+b^2$ . Hence

*Any formula deduced from the equation of the ellipse is changed to the corresponding formula for the hyperbola by merely changing  $+b^2$  to  $-b^2$ , or  $b$  to  $b\sqrt{-1}$ .*

The lengths  $r, r'$  of the focal radii for any point  $(x, y)$  are

$$r = \pm (ex - a), \quad r' = \pm (ex + a),$$

in which the upper signs hold for the right-hand branch, and the lower for the left.

156. A discussion of equation [40] leads to the following conclusions:

(i.) The curve cuts the axis of  $x$  at the two real points  $(a, 0)$  and  $(-a, 0)$ .

(ii.) The curve does not cut the axis of  $y$ . The imaginary intercepts are  $\pm b\sqrt{-1}$ .

(iii.) No part of the curve lies between the straight lines  $x = +a$  and  $x = -a$ .

(iv.) Outside these lines the curve extends without limit both to the right and to the left.

(v.) The greater the abscissa, the greater the ordinate.

(vi.) The curve is symmetrical with respect to the axis of  $x$ .

(vii.) The curve is symmetrical with respect to the axis of  $y$ .

(viii.) Every chord which passes through the centre is bisected by the centre. This explains why the point half way between the foci is called the centre.

The two distinct parts of the curve are called the *right-hand* and the *left-hand* branches.



157. An hyperbola whose transverse and conjugate axes are equal is called an **Equilateral Hyperbola**. Its equation is

$$x^2 - y^2 = a^2. \quad [41]$$

The equilateral hyperbola bears to the general hyperbola the same relation that the auxiliary circle bears to the ellipse.

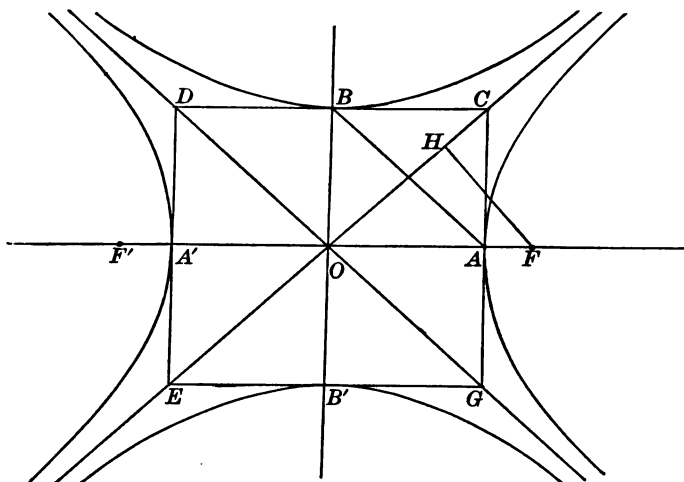


Fig. 70.

158. The hyperbola which has  $BB'$  for transverse axis, and  $AA'$  for conjugate axis, obviously holds the same relation to the axis of  $y$  that the hyperbola which has  $AA'$  for transverse axis and  $BB'$  for conjugate axis holds to the axis of  $x$ .

Therefore its equation is found by simply changing the signs of  $a^2$  and  $b^2$  in [40], and is

$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ or } \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1. \quad (1)$$

The two hyperbolas are said to be **Conjugate**.

159. The straight line  $y = mx$ , passing through the centre of the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ , meets the curve in two points, the abscissas of which are

$$x_1 = \frac{+ab}{\sqrt{b^2 - m^2a^2}}, \quad x_2 = \frac{-ab}{\sqrt{b^2 - m^2a^2}}.$$

Hence the points will be *real*, *imaginary*, or *situated at infinity*, as  $b^2 - m^2a^2$  is positive, negative, or zero; that is, as  $m^2$  is *less than*, *greater than*, or *equal to*  $\frac{b^2}{a^2}$ .

The same line,  $y = mx$ , will meet the conjugate hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$  in two points, whose abscissas are

$$x_1 = \frac{+ab}{\sqrt{m^2a^2 - b^2}}, \quad x_2 = \frac{-ab}{\sqrt{m^2a^2 - b^2}}.$$

Hence these points will be *imaginary*, *real*, or *situated at infinity*, as  $m^2$  is *less than*, *greater than*, or *equal to*  $\frac{b^2}{a^2}$ .

Whence

*If a straight line through the centre meet an hyperbola in imaginary points, it will meet the conjugate hyperbola in real points, and vice versa.*

160. An **Asymptote** is a straight line which passes through finite points, and meets a curve in *two* points at infinity.

We see from § 159 that the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

has two real asymptotes passing through the centre of the curve, and having for their equations  $y = +\frac{b}{a}x$  and  $y = -\frac{b}{a}x$ ; or, expressed in one equation,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0. \quad [42]$$

## Ex. 36.

What is the equation of an hyperbola, if

1. Transverse axis = 16, conjugate axis = 14?
2. Conjugate axis = 12, distance between foci = 13?
3. Distance between foci = twice the transverse axis?
4. Transverse axis = 8, one point is (10, 25)?
5. Distance between foci =  $2c$ , eccentricity =  $\sqrt{2}$ ?
6. Prove that the latus rectum of an hyperbola is equal to  $\frac{2b^2}{a}$ . Also  $2a : 2b :: 2b : \text{latus rectum}$ .
7. The equation of an hyperbola is  $9x^2 - 16y^2 = 144$ ; find the axes, distance between the foci, eccentricity, and latus rectum.
8. Write the equation of the hyperbola conjugate to the hyperbola  $9x^2 - 16y^2 = 144$ , and find its axes, distance between its foci, and its latus rectum.
9. If the vertex of an hyperbola bisects the distance from the centre to the focus, find the ratio of its axes.
10. Prove that the point  $(x, y)$  is *without*, *on*, or *within* the hyperbola, according as  $\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1$  is *negative*, *zero*, or *positive*.
11. Find the eccentricity of an equilateral hyperbola.
12. Determine the points common to the hyperbola  $25x^2 - 9y^2 = 225$ , and the straight line  $25x + 12y - 45 = 0$ .
13. The asymptotes of an hyperbola are the diagonals of the rectangle  $CDEG$  (Fig. 70, p. 172).
14. Find the foci and the asymptotes of the hyperbola  $16x^2 - 9y^2 = 144$ .

15. The asymptotes of an equilateral hyperbola are perpendicular to each other. Hence the equilateral hyperbola is also called the *rectangular* hyperbola.

16. An hyperbola and its conjugate have the same asymptotes.

17. Find the length of a perpendicular dropped from the focus to an asymptote.

18. Prove that the squares of any two ordinates of the hyperbola are to each other as the products of the segments into which they divide the transverse axis externally.

### TANGENTS AND NORMALS.

NOTE. The results stated in the following six sections may be established in the same way as the corresponding propositions in the ellipse, or the first five may be obtained by § 155.

161. The slope of the tangent at  $(x_1, y_1)$  is  $\frac{b^2 x_1}{a^2 y_1}$ , and its equation is

$$\frac{x_1 x}{a^2} - \frac{y_1 y}{b^2} = 1. \quad [43]$$

162. The equation of the normal at  $(x_1, y_1)$  is

$$y - y_1 = -\frac{a^2 y_1}{b^2 x_1} (x - x_1). \quad [44]$$

163. The subtangent  $= \frac{x_1^2 - a^2}{x_1}$ , the subnormal  $= \frac{b^2 x_1}{a^2}$ .

164. The straight line whose equation is  $y = mx \pm \sqrt{m^2 a^2 - b^2}$  is a tangent for all values of  $m$  (§ 135).

165. The equation of the director circle of an hyperbola is  $x^2 + y^2 = a^2 - b^2$  (§ 136).

166. The tangent and the normal at any point of an hyperbola bisect the angles formed by the focal radii of the point (§ 133).

## Ex. 37.

1. Find the equations of tangent and normal to the hyperbola  $16x^2 - 9y^2 = 112$  at the point of contact  $(4, 4)$ . Also find the lengths of the subtangent and the subnormal.

2. Show that in an equilateral hyperbola the subnormal is equal to the abscissa of the point of contact.

3. The equations of the tangent and the normal at a point of an equilateral hyperbola are  $5x - 4y = 9$ ,  $4x + 5y = 40$ . What is the equation of the hyperbola, and what are the co-ordinates of the point of contact?

4. For what points of an hyperbola is the subtangent equal to the subnormal?

5. To draw a tangent and a normal to an hyperbola at a given point of the curve.

6. If an ellipse and an hyperbola have the same foci, prove that the tangents to the two curves drawn at their points of intersection are perpendicular to each other.

7. Prove that the asymptotes of an hyperbola are the limiting positions of tangents to the infinite branches.

8. Prove that the length of a normal in an equilateral hyperbola is equal to the distance of the point of contact from the centre.

9. Find the distance from the origin to the tangent through the end of the latus rectum of the equilateral hyperbola  $x^2 - y^2 = a^2$ .

10. What condition must be satisfied in order that the straight line  $\frac{x}{m} + \frac{y}{n} = 1$  may touch the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ ?

11. When is the director circle of an hyperbola imaginary?

12. Find the locus of the foot of a perpendicular dropped from the focus of an hyperbola to a tangent.

## Ex. 38. (Review.)

1. The ordinate through the focus of an hyperbola, produced, cuts the asymptotes in  $P$  and  $Q$ . Find  $PQ$  and the distances of  $P$  and  $Q$  from the centre.

2. In the hyperbola  $9x^2 - 16y^2 = 144$  what are the focal radii of the points whose common abscissa is 8? What other points have equal focal radii?

3. What relation exists between the *sum* of the focal radii of a point of an hyperbola and the abscissa of the point?

4. Prove that in the equilateral hyperbola every ordinate is a mean proportional between the distances of its foot from the vertices of the curve. Hence find a method of constructing an equilateral hyperbola when the axes are given.

5. In the equilateral hyperbola the distance of a point from the centre is a mean proportional between its focal radii.

6. In the equilateral hyperbola the bisectors of the angles formed by lines drawn from the vertices to any point of the curve are parallel to the asymptotes.

7. If  $e, e'$  are the eccentricities of two conjugate hyperbolas,

$$\frac{1}{e^2} + \frac{1}{e'^2} = 1.$$

8. Through the positive vertex of an hyperbola a tangent is drawn. In what points does it cut the conjugate hyperbola?

9. The sum of the reciprocals of two focal chords perpendicular to each other is constant.

10. Through the foot of the ordinate of a point in an equilateral hyperbola a tangent is drawn to the circle described upon the transverse axis as diameter. What relation exists between the lengths of this tangent and the ordinate of the point?

11. In an equilateral hyperbola find the equations of tangents drawn from the positive end of the conjugate axis.

12. From what point in the conjugate axis of an hyperbola must tangents be drawn in order that they may be perpendicular to each other?

13. What condition must be satisfied that a square may be constructed whose sides shall be parallel to the axes of an hyperbola and whose vertices shall lie in the curve?

14. Find the equation of the chord of the hyperbola  $16x^2 - 9y^2 = 144$  which is bisected at the point  $(12, 3)$ .

15. Find the equation of a tangent to the hyperbola  $16x^2 - 9y^2 = 144$  parallel to the line  $y = 4x - 3$ .

16. Find the product of the two perpendiculars let fall from any point of any hyperbola upon the asymptote.

17. A chord of an hyperbola which touches the conjugate hyperbola is bisected at the point of contact.

### SUPPLEMENTARY PROPOSITIONS.

NOTE. Many of the following propositions are closely analogous to propositions already established for the ellipse; hence the proofs are omitted, and references given to the chapter on the ellipse.

167. *Two distinct, two coincident, or no tangents can be drawn to an hyperbola through any point  $(h, k)$ , according as the point is without, on, or within this curve (§ 137).*

168. *The equation of the chord of contact of the two tangents drawn from the external point  $(h, k)$  to the hyperbola*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad \text{is} \quad \frac{hx}{a^2} - \frac{ky}{b^2} = 1. \quad (\S 138)$$

169. *The equation of the polar of the pole  $(h, k)$  with regard to the hyperbola is*

$$\frac{hx}{a^2} - \frac{ky}{b^2} = 1. \quad (\S 139)$$

The tangent and chord of contact are particular cases of the polar, and § 74 holds true in the hyperbola.

170. *The equation of a diameter of an hyperbola is*

$$y = \frac{b^2}{a^2 m} x, \quad (\S 141)$$

*in which  $m$  is the slope of its chords.*

171. If  $m'$  is the slope of the diameter bisecting the chords parallel to the diameter  $y = mx$ , then (§ 142)

$$mm' = \frac{b^2}{a^2}. \quad [45]$$

Since  $m$  and  $m'$  are alike involved in [45], it follows that

*If one diameter bisects all chords parallel to another, the second diameter will bisect all chords parallel to the first.*

Two diameters drawn so that each bisects all chords parallel to the other are called **Conjugate Diameters**.

172. From [45], the slopes of two conjugate diameters must agree in sign; hence

*Two conjugate diameters of an hyperbola lie on the same side of the conjugate axis, and their included angle is acute.*

Also, if  $m$  in absolute magnitude is less than  $\frac{b}{a}$ , then  $m'$  must be greater than  $\frac{b}{a}$ . But the slope of the asymptotes is equal to  $\pm \frac{b}{a}$ . Therefore

*Two conjugate diameters lie on opposite sides of the asymptote in the same quadrant; and of two conjugate diameters, one meets the curve in real points and the other in imaginary points (§ 159).*



173. The *length* of a diameter which meets the hyperbola in real points is the length of the chord between these points.

If a diameter meets the hyperbola in imaginary points, that is, does not meet it at all, it will meet the conjugate hyperbola in real points (§ 159); and its length is the length of the chord between these points. But from § 159 we know that if a diameter meet one of the hyperbolas in the imaginary point  $(h\sqrt{-1}, k\sqrt{-1})$ , it will meet the other in the real point  $(h, k)$ ; hence the length of the semi-diameter, which is  $\sqrt{h^2 + k^2}$ , is known from the imaginary co-ordinates of intersection.

174. The equations of an hyperbola and its conjugate differ only in the signs of  $a^2$  and  $b^2$ . But this interchange of signs does not effect the equation

$$mm' = \frac{b^2}{a^2}.$$

Therefore, if two diameters are conjugate with respect to one of two conjugate hyperbolas, they will be conjugate with respect to the other.

Thus, let  $POP'$  and  $QOQ'$  (Fig. 71) be two conjugate diameters. Then  $POP'$  bisects all chords parallel to  $QOQ'$  that lie *within* the branches of the original hyperbola and *between* the branches of the conjugate hyperbola; and  $QOQ'$  bisects all chords parallel to  $POP'$  that lie *within* the branches of the conjugate hyperbola and *between* the branches of the original hyperbola.

From the above theorem it follows immediately that

*If a straight line meet each of two conjugate hyperbolas in two real points, the two portions of the line contained between the hyperbolas are equal (thus,  $BD = B'D'$ , Fig. 71).*

175. The tangent drawn through the end of a diameter is parallel to the conjugate diameter (§ 143).

176. Having given the end  $(x_1, y_1)$  of a diameter, to find the end  $(x_2, y_2)$  of the conjugate diameter.

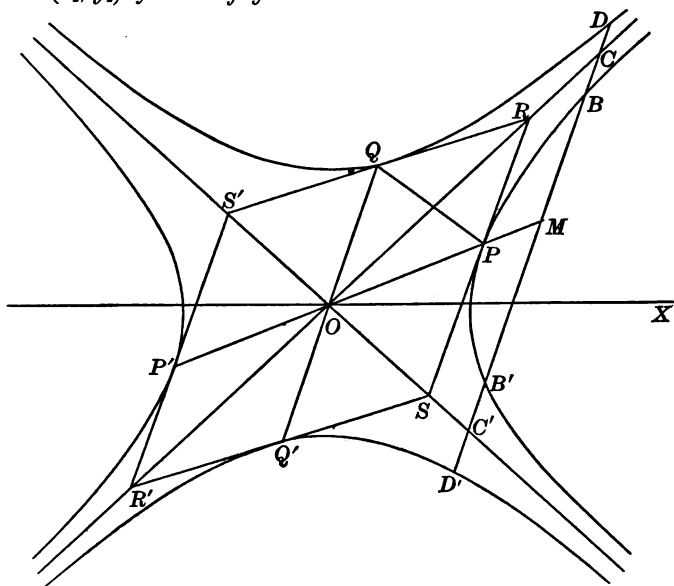


Fig. 71.

Let  $(x_1, y_1)$  be on the given hyperbola, then  $(x_2, y_2)$  is on the conjugate. The slope of the tangent at  $(x_1, y_1)$  is  $\frac{b^2 x_1}{a^2 y_1}$ ; hence the equation of the diameter conjugate to the diameter through  $(x_1, y_1)$  is

$$y = \frac{b^2 x_1}{a^2 y_1} x. \quad (1)$$

Now  $(x_2, y_2)$  is on the diameter (1) and also on the conjugate hyperbola; hence we have

$$y_2 = \frac{b^2 x_1}{a^2 y_1} x_2, \quad \frac{x_2^2}{a^2} - \frac{y_2^2}{b^2} = -1. \quad (2)$$

Solving equations (2) for  $x_2$  and  $y_2$ , we obtain

$$x_2 = \pm \frac{a}{b} y_1, \quad y_2 = \pm \frac{b}{a} x_1.$$

The positive signs belong to one end, and the negative signs to the other end, of the conjugate diameter.

177. *To find the equation of an hyperbola referred to any pair of conjugate diameters as axes of co-ordinates.*

From the symmetry of the curve with respect to each of the new axes, the required equation must be of the form

$$Ax^2 + By^2 = C.$$

Denoting the intercepts of the curve on the new axes by  $a'$  and  $b' \sqrt{-1}$  (§ 172), we obtain

$$A = \frac{C}{a'^2}, \quad B = -\frac{C}{b'^2}.$$

Whence 
$$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 1 \quad (1)$$

is the required equation, in which  $a'$  and  $b'$  are semi-conjugate diameters.

Since the form of equation (1) is the same as that of the equation referred to the axes of the curve, it follows that all formulas which have been obtained without assuming the axes of co-ordinates to be at right angles to each other hold good when the axes of co-ordinates are any two conjugate diameters. For example, the equation of the asymptotes of the hyperbola represented by equation (1) is

$$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 0, \quad (2)$$

and the equation of the tangent is

$$\frac{x_1 x}{a'^2} - \frac{y_1 y}{b'^2} = 1.$$

178. *The tangents through the ends of two conjugate diameters meet in the asymptotes.*

The equations of these tangents referred to the conjugate diameters are

$$x = \pm a', \quad y = \pm b'.$$

Hence their intersections are  $(a', b')$ ,  $(a', -b')$ ,  $(-a', b')$ , and  $(-a', -b')$ . But these points evidently lie upon the asymptotes, or the locus of (2) in § 177.

179. If  $\theta$  denote the angle formed by two conjugate semi-diameters, and  $a'$  and  $b'$ , their lengths, then  $\sin \theta = \frac{ab}{a'b'}$ .

Substituting  $b\sqrt{-1}$  for  $b$ , and  $b'\sqrt{-1}$  for  $b'$  in (1) of § 147, and cancelling the imaginary factors, we obtain the above result.

COR. 1. Since  $4a'b' \sin \theta = 4ab$ , the parallelogram  $SRS'R'$  (Fig. 71) equals the rectangle on the axes of the curve.

COR. 2. The length of the perpendicular from  $O$  upon the tangent  $SPR = OP \sin OPS = a' \sin \theta = \frac{ab}{b'}$ .

COR. 3. From §§ 145, 155, 177, we have

$$a'^2 - b'^2 = a^2 - b^2.$$

180. If a straight line cut an hyperbola and its asymptotes, the portions of the line intercepted between the curve and its asymptotes are equal.

Let  $CC'$  (Fig. 72) be the line meeting the asymptotes in  $C$ ,  $C'$  and the curve in  $B$ ,  $B'$ , and let the equation of the line be

$$y = mx + c. \quad (1)$$

Let  $M$  be the middle point of the chord  $BB'$ ; then (§ 170) the equation of the diameter through  $m$  is

$$y = \frac{b^2x}{a^2m}. \quad (2)$$

By combining equation (1) with the equations of the asymptotes, we obtain the co-ordinates of the points  $C$  and  $C'$ ; taking the half-sum of these values, we get for the co-ordinates of the point half way between  $C$  and  $C'$  the values

$$x = \frac{ma^2c}{b^2 - m^2a^2}, \quad y = \frac{b^2c}{b^2 - m^2a^2}.$$

These values satisfy equation (2); therefore the point half way between  $C$  and  $C'$  coincides with  $M$ ; therefore  $MC = MC'$ . And since  $MB = MB'$ , therefore  $BC = B'C'$ .

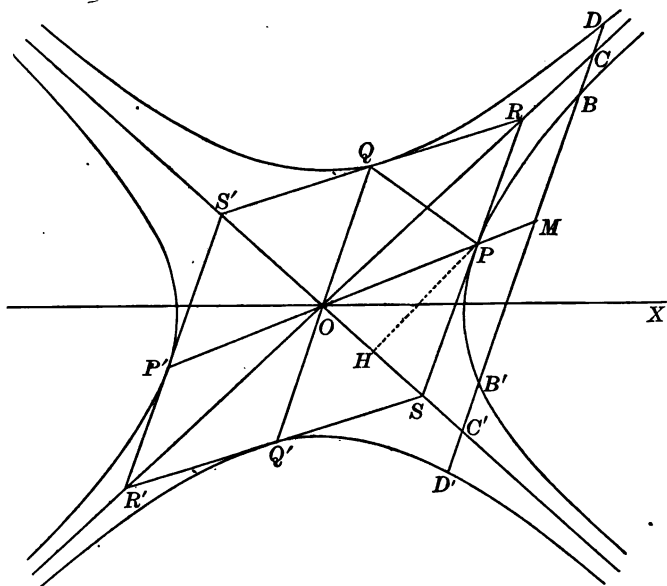


Fig. 72.

**COR.** Let  $CC'$  be moved parallel to itself till it becomes a tangent at  $P$ , meeting the asymptotes in  $R, S$ ; then the points  $B, B'$  coincide at  $P$ , and we have  $PR = PS$ . Hence

*The portion of a tangent intercepted by the asymptotes is bisected by the point of contact.*

**181.** The following method of showing that an hyperbola has asymptotes, and finding their equations, is more general than the method given in §§ 159, 160.

The abscissas of the points where the straight line  $y = mx + c$  meets an hyperbola are found by solving the equation

$$\frac{x^2}{a^2} - \frac{(mx + c)^2}{b^2} = 1,$$

or 
$$\frac{b^2 - m^2 a^2}{a^2 b^2} x^2 - \frac{2mc}{b^2} x = \frac{b^2 + c^2}{b^2}. \quad (1)$$

Now from algebra we know that as the coefficients of  $x^2$  and  $x$  in (1) approach zero, both roots of (1) increase without limit. Hence each root becomes infinity when

$$b^2 - m^2 a^2 = 0, \text{ and } 2mc = 0,$$

or when 
$$m = \pm \frac{b}{a}, \text{ and } c = 0.$$

Therefore,  $y = \pm \frac{b}{a}x$  are asymptotes to the hyperbola.

If only  $b^2 - m^2 a^2 = 0$ , then  $m = \pm \frac{b}{a}$ , the line is parallel to an asymptote, and one root of (1) is infinity, while the other is 
$$-\frac{b^2 + c^2}{2mc}.$$

Hence, a right line parallel to an asymptote meets the hyperbola in only one finite point.

**182.** To find the equation of an hyperbola referred to the asymptotes as axes of co-ordinates.

Let the lines  $OB, OC$  (Fig. 73) be the asymptotes,  $A$  the vertex of the curve, and let the angle  $AOC = \alpha$ .

Let the co-ordinates of any point  $P$  of the curve be  $x, y$  when referred to the axes of the curve, and  $x', y'$  when referred to  $OB, OC$  as axes of co-ordinates.

Draw  $PM \perp$  to  $OA$ ,  $PN \parallel$  to  $OC$ ; then

$$x = ON \cos \alpha + NP \cos \alpha = (x' + y') \cos \alpha,$$

$$y = NP \sin \alpha - ON \sin \alpha = (y' - x') \sin \alpha.$$

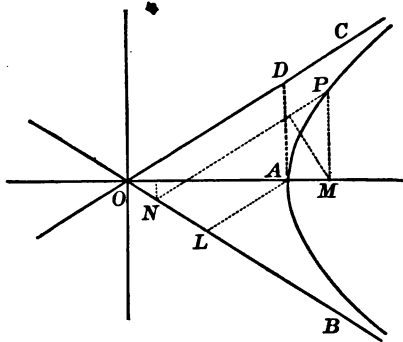


Fig. 73.

Hence, by substituting in [40], we obtain

$$\frac{(x' + y')^2 \cos^2 \alpha}{a^2} - \frac{(y' - x')^2 \sin^2 \alpha}{b^2} = 1.$$

But

$$\sin \alpha = \frac{AD}{OD} = \frac{b}{\sqrt{a^2 + b^2}},$$

$$\cos \alpha = \frac{OA}{OD} = \frac{a}{\sqrt{a^2 + b^2}}.$$

Substituting these values, and dropping accents, we have

$$4xy = a^2 + b^2. \quad [46]$$

COR. 1. The equation of the conjugate hyperbola is

$$4xy = - (a^2 + b^2).$$

COR. 2.  $\sin COB = \sin 2\alpha = 2 \sin \alpha \cos \alpha = \frac{2ab}{a^2 + b^2}.$

If  $a = b$ ,  $\sin COB = 1$ ; therefore  $COB = \frac{1}{2}\pi$ .

COR. 3. Let  $(x_1, y_1)$  denote  $P$  (Fig. 72), referred to the asymptotes; then

$$OS \times OR = 2OH \times 2HP = 4x_1y_1 = a^2 + b^2.$$

That is, *the product of the intercepts of a tangent upon the asymptotes is equal to the sum of the squares of the semi-axes.*

COR. 4. In Fig. 72, the area of the triangle  $ROS$  equals

$$\frac{1}{2} OS \times OR \sin ROS = \frac{1}{2} (a^2 + b^2) \frac{2ab}{a^2 + b^2} = ab.$$

That is, *the area of the triangle formed by any tangent and the asymptotes is equal to the product of the semi-axes.*

183. The polar of the focus  $(ae, 0)$  is

$$x = \frac{a}{e} = \frac{a^2}{ae}.$$

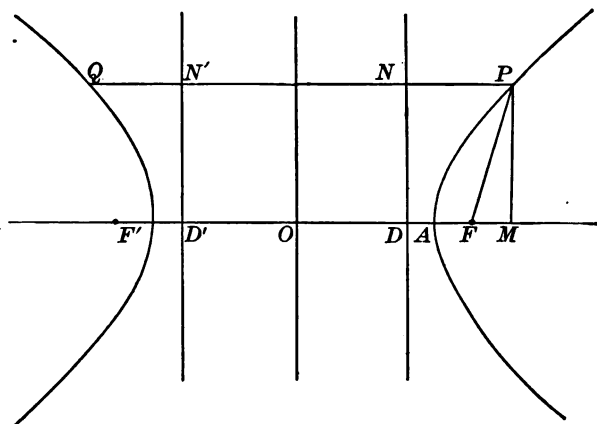


Fig. 74.

Hence, if  $OD$  be taken so that

$$OF : OA = OA : OD,$$

then  $DN$  perpendicular to  $OF$  is the polar of  $F$ , and is called a **Directrix** of the hyperbola. In like manner we may construct  $D'N'$ , or the directrix corresponding to the focus  $F'$ .

COR. As in § 150 we may prove that

$$e = \frac{PF}{PN}.$$



Whence the hyperbola may be defined as

*The locus of a point whose distances from a fixed point and a fixed straight line bear a constant ratio greater than unity.*

184. To find the polar equation of an hyperbola, the left-hand focus being taken as pole.

If  $x$  be reckoned from the centre, and we write

$$\rho = ex + a, \quad (1)$$

$\rho$  will be positive or negative according as the point is on the right or left hand branch.

$$\text{Now } x = \rho \cos \theta - c = \rho \cos \theta - ae.$$

Whence, by substitution and reduction,

$$\rho = \frac{a(e^2 - 1)}{e \cos \theta - 1}. \quad [47]$$

From (1) we know that a point is on the right or left hand branch, according as  $\rho$  in [47] is positive or negative; that is, according as  $\cos \theta > \text{or } < \frac{1}{e}$ .

$$\text{For } \theta = 0, \rho = ae + a = F'O + OA = F'A \text{ (Fig. 70).}$$

For  $e \cos \theta - 1 = 0$ , or  $\theta = \cos^{-1} \frac{1}{e}$ ,  $\rho = \infty$ , as it should, since in this case the radius vector is parallel to the asymptote.

$$\text{For } \theta = \frac{1}{2}\pi, \rho = -a(e^2 - 1) = -\text{semi-latus rectum.}$$

$$\text{For } \theta = \pi, \rho = a - ae = -F'A'.$$

#### Ex. 39.

1. What is the polar of the point  $(-9, 7)$  with respect to the hyperbola  $7x^2 - 12y^2 = 112$ ?

2. Find the equations of the directrices of an hyperbola.

3. Find the angle formed by a focal chord and the line which joins its pole to the focus.

4. Find the pole of the line  $Ax + By + C = 0$  with respect to an hyperbola.

5. Find the polar of the right-hand vertex of an hyperbola with respect to the conjugate hyperbola.

6. Find the distance from the centre of an hyperbola to the point where the directrix cuts the asymptote.

7. If  $(x_1, y_1)$  and  $(x_2, y_2)$  are the ends of two conjugate diameters, then

$$\frac{x_1 x_2}{a^2} - \frac{y_1 y_2}{b^2} = 0.$$

8. The equation of a diameter in the hyperbola  $25x^2 - 16y^2 = 400$  is  $3y = x$ . Find the equation of the conjugate diameter.

9. In the hyperbola  $49x^2 - 4y^2 = 196$ , find the equation of that chord which is bisected at the point  $(5, 3)$ .

10. Find the length of the semi-diameter conjugate to the diameter  $y = 3x$  in the hyperbola  $9x^2 - 4y^2 = 36$ .

11. Two tangents to a hyperbola at right angles intersect on the circle

$$x^2 + y^2 = a^2 - b^2.$$

12. Tangents at the extremities of any chord of an hyperbola intersect on the diameter which bisects that chord.

13. Prove that  $PQ$  (Fig. 71) is parallel to one asymptote and bisected by the other.

14. An asymptote is its own conjugate diameter.

15. The conjugate diameters of an equilateral hyperbola are equal.

16. Having given two conjugate diameters in length and position, to find by construction the asymptotes and the axes.

17. To draw a tangent to an hyperbola from a given point.

18. Find the equation of a tangent at any point  $(x_1, y_1)$  of the hyperbola  $4xy = a^2 + b^2$ .

19. Find the equation of an hyperbola, taking as the axis of  $y$  (i.) the tangent through the left-hand vertex ;  
(ii.) the tangent through the right-hand vertex.

20. Find the polar equation of an hyperbola, taking the right-hand focus as pole.

21. Find the polar equation of an hyperbola, taking the centre as pole.

22. To find the centre of a given hyperbola.

23. The distance from a fixed point to a fixed straight line is 10. Find the locus of a point which moves so that its distance from the fixed point is always twice its distance from the fixed line.

24. Show that the locus of  $x^2 - 4y^2 - 2x - 16y - 19 = 0$  is an hyperbola ; find its centre and axes.

25. If  $A$  and  $B$  have unlike signs, prove that the locus of  $Ax^2 + By^2 + Dx + Ey + F = 0$  is in general an hyperbola whose axes are parallel to the co-ordinate axes ; and determine its semi-axes.

26. Through the point  $(-4, 7)$  a straight line is drawn to meet the axes of co-ordinates, and then revolved about this point. Find the locus of the point midway between the axes.

27. A straight line has its ends in two fixed perpendicular lines, and forms with them a triangle of constant area  $a^2$ . Find the locus of the middle point of the line.

28. The base  $a$  of a triangle is fixed in length and position, and the vertex so moves that one of the base angles is always double the other. Find the locus of the vertex.

## CHAPTER VIII.

### LOCI OF THE SECOND ORDER.

**185.** The loci represented by equations of the second degree that are not of the first order are called **Loci of the Second Order.**

In the preceding chapters we have seen that the circle, parabola, ellipse, and hyperbola are loci of the second order. We now propose to inquire whether there are other loci of the second order besides the four curves just named; in other words, to find what loci may be represented by equations of the second degree.

For this purpose we shall write the general equation of the second degree in the form

$$Ax^2 + By^2 + Cxy + Dx + Ey + F = 0, \quad (1)$$

and shall assume that the axes of co-ordinates are rectangular. This assumption will in nowise diminish the generality of our conclusions; for if the axes were oblique, we could change them to rectangular axes, and this change would not alter the degree of the equation or the nature of the locus which it represents (§ 91).

**186.** *To find the condition that the general equation of the second degree may represent two loci of the first order.*

To do this let us solve (1) with respect to one of the variables. Choosing  $y$  for this purpose, we obtain

$$y = -\frac{Cx + E}{2B} \pm \frac{1}{2B} \sqrt{Lx^2 + Mx + N}, \quad (2)$$

where  $L = C^2 - 4AB$ ,  $M = 2(CE - 2BD)$ ,  $N = E^2 - 4BF$ .

If  $Lx^2 + Mx + N$  be a perfect square, then the locus of (2), or (1), will be two loci of the first order.

Now, from Algebra, we know that the condition that  $Lx^2 + Mx + N$  should be a perfect square is

$$M^2 - 4LN = 0;$$

or, substituting the values of  $L$ ,  $M$ , and  $N$ , we have

$$(CE - 2BD)^2 - (C^2 - 4AB)(E^2 - 4BF) = 0,$$

$$\text{or} \quad F(C^2 - 4AB) + AE^2 + BD^2 - CDE = 0. \quad (3)$$

The quantity on the left-hand side of equation (3) is usually denoted by  $\Delta$ , and is called the **Discriminant** of equation (1).

This same result was obtained by a more general method in § 57; hence

*Whenever  $\Delta = 0$ , equation (1) represents two loci of the first order. These loci may be readily determined by resolving (1) into two simple equations in  $x$  and  $y$ .*

#### CENTRAL CURVES. $\Sigma$ NOT ZERO.

**187.** A centre of a curve is a point that bisects every chord passing through it. Loci are classified as **Central** and **Non-Central**, according as they have or have not a definite centre. The circle, ellipse, and hyperbola belong to the first class, and the parabola to the second.

**188.** *To find the equation of the central loci represented by equation (1) referred to their centre.*

To do this let us change the origin to the point  $(h, k)$ , and then so choose the values of  $h$  and  $k$  that the terms involving the first powers of  $x$  and  $y$  will vanish. Making the change by substituting in (1)  $x+h$  for  $x$ , and  $y+k$  for  $y$ , we find that the coefficients  $A$ ,  $B$ , and  $C$  remain unaltered, and we may write the transformed equation

$$Ax^2 + By^2 + Cxy + D'x + E'y = R. \quad (4)$$

$$\begin{aligned} \text{where} \quad D' &= 2Ah + Ck + D, \\ E' &= 2Bk + Ch + E, \\ R &= -[Ah^2 + Bk^2 + Chk + Dh + Ek + F]. \end{aligned}$$

The values of  $h$  and  $k$  which will make  $D'$  and  $E'$  vanish are evidently found by solving the equations

$$\begin{aligned} 2Ah + Ck + D &= 0, \\ 2Bk + Ch + E &= 0, \end{aligned}$$

$$\text{and are} \quad h = \frac{CE - 2BD}{4AB - C^2}, \quad k = \frac{CD - 2AE}{4AB - C^2}.$$

If  $4AB - C^2$ , denoted by  $\Sigma$ , is not zero, these values of  $h$  and  $k$  are finite and single, and equation (4) may be written

$$Ax^2 + By^2 + Cxy = R. \quad (5)$$

From the form of (5) we see that if  $(x, y)$  be a point in its locus, so also is  $(-x, -y)$ ; that is, the new origin  $(h, k)$  is the centre of the locus. Hence

*When  $\Sigma$  is not zero, equation (1) can be reduced to the form of (5), and represents central curves.*

When, however,  $\Sigma = 0$ , the values of  $h$  and  $k$  become infinite or indeterminate, and the locus of (1) has no definite centre. Hence

*When  $\Sigma = 0$ , (1) cannot be reduced to the form of (5), and represents non-central curves.*

The value of  $R$  can be reduced to the following useful form, which shows also that  $R$  and  $\Delta$  vanish together.

$$\begin{aligned} R &= -[Ah^2 + Bk^2 + Chk + Dh + Ek + F] \\ &= -\frac{1}{2}[(2Ah + Ck + D)h + (2Bk + Ch + E)k + Dh + Ek + 2F] \\ &= -\frac{1}{2}(D'h + E'k + Dh + Ek + 2F) \\ &= -\frac{1}{2}(Dh + Ek + 2F) \\ &= -\frac{1}{2} \frac{2BD^2 - CDE + 2AE^2 - CDE + 2F(C^2 - 4AB)}{C^2 - 4AB} \\ &= \frac{\Delta}{\Sigma}. \end{aligned}$$

189. To reduce (5) to a known form by causing the term in  $xy$  to disappear.

For this purpose we change the direction of the axes through an angle  $\theta$ , keeping the origin unaltered, and then determine the value of  $\theta$  by putting the new term which involves  $xy$  equal to zero.

The change is made by substituting for  $x$  and  $y$ , in equation (5), the respective values (§ 86),

$$\begin{aligned}x \cos \theta - y \sin \theta, \\ x \sin \theta + y \cos \theta;\end{aligned}$$

and equation (5) now becomes

$$Px^2 + Qy^2 + C'xy = R,$$

where

$$P = A \cos^2 \theta + B \sin^2 \theta + C \sin \theta \cos \theta, \quad (6)$$

$$Q = A \sin^2 \theta + B \cos^2 \theta - C \sin \theta \cos \theta, \quad (7)$$

$$C' = 2(B - A) \sin \theta \cos \theta + C(\cos^2 \theta - \sin^2 \theta). \quad (8)$$

Putting  $C' = 0$ , we obtain, by Trigonometry, § 33,

$$(A - B) \sin 2\theta - C \cos 2\theta = 0, \quad (9)$$

$$\text{or} \quad \tan 2\theta = \frac{C}{A - B}. \quad (10)$$

Since any real number, positive or negative, is the tangent of some angle between zero and  $\pi$ , equation (10) is satisfied by some value of  $\theta$  between zero and  $\frac{1}{2}\pi$ . In what follows, we shall use this simplest root of (10).

By this transformation, equation (5) is reduced to the form

$$Px^2 + Qy^2 = R, \quad (11)$$

of which the discussion will be found in the next section.

COR. 1. The values of  $P$  and  $Q$  in terms of  $A$ ,  $B$ ,  $C$  may be found as follows:

From (6) and (7), by addition and subtraction,

$$P + Q = A + B, \quad (12)$$

$$P - Q = (A - B) \cos 2\theta + C \sin 2\theta. \quad (13)$$

Equation (9) may be written

$$0 = (A - B) \sin 2\theta - C \cos 2\theta. \quad (14)$$

Adding the squares of (13) and (14), we have

$$(P - Q)^2 = (A - B)^2 + C^2, \quad (15)$$

$$\text{or} \quad P - Q = \pm \sqrt{(A - B)^2 + C^2}. \quad (16)$$

Whence, from (12) and (16),

$$P = \frac{1}{2}[A + B \pm \sqrt{(A - B)^2 + C^2}], \quad (17)$$

$$Q = \frac{1}{2}[A + B \mp \sqrt{(A - B)^2 + C^2}]. \quad (18)$$

These values of  $P$  and  $Q$  are evidently always real.

COR. 2. By squaring (12) and subtracting (15), we obtain

$$4PQ = 4AB - C^2 = \Sigma. \quad (19)$$

Hence,  $P$  and  $Q$  have like or unlike signs, according as  $\Sigma$  is positive or negative.

COR. 3. In applying formulas (17) and (18), the question arises which sign before the radical should be used. If in (13) we substitute for  $\cos 2\theta$ , its value obtained from (14), we have

$$P - Q = \frac{[(A - B)^2 + C^2] \sin 2\theta}{C}.$$

Since the numerator of the fraction is always positive,  $P - Q$  must have the same sign as  $C$ ; that is, the upper or lower sign in (16) must be taken according as  $C$  is positive or negative.

Hence, *the upper or lower signs in (17) and (18) are to be taken according as  $C$  is positive or negative.*

190. The nature of the locus represented by equation (11) depends upon the signs of  $P$ ,  $Q$ , and  $R$ . There are two groups of cases, according as  $\Sigma$  is positive or negative, and three cases in each group.



*Group 1.  $\Sigma$  Positive.*

In this group,  $P$  and  $Q$  must, by (19), agree in sign.

CASE 1. If  $R$  agrees in sign with  $P$  and  $Q$ , then, by § 124, the locus is an ellipse whose semi-axes are  $\sqrt{\frac{R}{P}}$  and  $\sqrt{\frac{R}{Q}}$ .

If  $P = Q$ , the locus is a circle.

CASE 2. If  $R$  differs from  $P$  and  $Q$  in sign, no real values of  $x$  and  $y$  will satisfy (11), so that no real locus exists.

CASE 3. If  $R = 0$ , the locus is the single point  $(0, 0)$ .

*Group 2.  $\Sigma$  Negative.*

In this group,  $P$  and  $Q$ , by (19), must have unlike signs.

CASE 1. If  $R$  agrees in sign with  $P$ , we may, by division (and by changing the signs of all the terms if necessary), put equation (11) into the form of equation [40], page 171. Therefore the locus is an *hyperbola*, with its transverse axis on the axis of  $x$ , and having for semi-axes

$$a = \sqrt{\frac{R}{P}}, \quad b = \sqrt{\frac{R}{-Q}}.$$

CASE 2. If  $R$  agrees in sign with  $Q$ , we may, by division (and by change of signs if necessary), put equation (11) into the form of equation (1), page 172. Therefore the locus is an *hyperbola*, with its transverse axis on the axis of  $y$ .

CASE 3. If  $R = 0$ , the locus consists of *two straight lines*, intersecting at the origin, and having for their equations

$$y = \pm \sqrt{\frac{P}{-Q}} x.$$

NON-CENTRAL CURVES.  $\Sigma = 0$ .

191. To determine the locus of (1) when  $\Delta$  and  $\Sigma$  are both zero.

If  $\Delta = \Sigma = 0$ , then from the first form of  $\Delta$  in § 186 we must have

$$CE - 2BD = 0. \quad (20)$$

Hence  $L = M = 0$  (§ 186)

and (2) becomes

$$2By + Cx + E \pm \sqrt{E^2 - 4BF} = 0, \quad (21)$$

which represents two parallel straight lines, two coincident straight lines, or no locus, according as  $E^2 - 4BF > = < 0$ .

When  $4AB - C^2 = 0$ , if  $C$  is not zero, neither  $A$  nor  $B$  can be zero; if  $C = 0$ , either  $A$  or  $B$  must be zero, but both cannot be, since if  $A = B = C = 0$ , (1) is no longer of the second degree.

When  $C = B = 0$ , by solving (1) for  $x$ , and introducing the above condition, we should obtain instead of (21) its corresponding equation

$$2Ax + Cy + D \pm \sqrt{D^2 - 4AF} = 0, \quad (22)$$

whose locus is also parallel straight lines. Hence,

*When  $\Delta$  and  $\Sigma$  are both zero, equation (1) represents two parallel straight lines, two coincident straight lines, or no locus.*

COR. Eliminating  $B$  between  $\Sigma = 0$  and (20), we obtain

$$CD - 2AE = 0. \quad (23)$$

In like manner (20) follows from  $\Sigma = 0$  and (23).

From these results, and the values of  $h$  and  $k$ , we learn that

(i.) When  $\Delta = \Sigma = 0$ ,  $h$  and  $k$  are both indeterminate, and conversely.

(ii.) The values of  $h$  and  $k$  are indeterminate together.

192. To determine the locus of (1) when  $\Sigma$  is zero and  $\Delta$  is not zero.

We simplify (1) by first making the term in  $xy$  to disappear by proceeding exactly as in § 189; that is, by turning the axes

through an angle  $\theta$ , the value of which is determined by the equation

$$\tan 2\theta = \frac{C}{A - B}. \quad (24)$$

If  $P, Q, U, V$  represent the new coefficients of  $x^2, y^2, x, y$ , respectively,  $P$  and  $Q$  will have values identical with those of  $P$  and  $Q$  given in § 189, and

$$U = D \cos \theta + E \sin \theta, \quad (25)$$

$$V = -D \sin \theta + E \cos \theta. \quad (26)$$

Since  $C^2 = 4AB$ , from (17) and (18) we have, when  $C$  is positive,

$$P = A + B, \quad Q = 0.$$

When  $C$  is negative,

$$P = 0, \quad Q = A + B,$$

and (1) assumes the form

$$Qy^2 + Ux + Vy + F = 0. \quad (27)$$

To further simplify, we divide by  $Q$ , and obtain

$$y^2 + \frac{U}{Q}x + \frac{V}{Q}y + \frac{F}{Q} = 0,$$

or 
$$y^2 + \frac{V}{Q}y + \frac{V^2}{4Q^2} + \frac{U}{Q}\left(x + \frac{F}{U} - \frac{V^2}{4QU}\right) = 0,$$

or 
$$\left(y + \frac{V}{2Q}\right)^2 = -\frac{U}{Q}\left(x + \frac{4QF - V^2}{4UQ}\right). \quad (28)$$

If we now take as a new origin the point

$$\left(-\frac{4QF - V^2}{4UQ}, -\frac{V}{2Q}\right),$$

equation (28) becomes

$$y^2 = -\frac{U}{Q}x,$$

which represents a parabola whose axis coincides with the axis of  $x$ , and which is situated on the *positive* or the *negative* side of the new origin, according as  $U$  and  $Q$  are *unlike* or *like* in sign (§ 94).

The vertex of the parabola is the new origin, and the parameter is equal to the coefficient of  $x$  in the equation of the curve.

This last transformation is possible except when  $U=0$ ; but, when  $U=0$ , (27) evidently represents parallel straight lines.

Suppose that  $C$  is positive. Then the general equation becomes

$$Px^2 + Ux + Vy + F = 0. \quad (29)$$

And this, by changing the origin to the point

$$\left( -\frac{4PF-U^2}{4VP}, -\frac{U}{2P} \right)$$

becomes  $x^2 = -\frac{V}{P}y$ .

This represents a *parabola* having the axis of  $y$  for its axis, and placed on the *positive* or the *negative* side of the new origin, according as  $V$  and  $P$  are *unlike* or *like* in sign.

It should be noted that the value of  $P$  or  $Q$ , when not zero, is  $A+B$ . The values of  $U$  and  $V^*$  can be found from (25) and (26).

\* We may obtain the values of  $U$  and  $V$  in terms of the original coefficients, as follows:

From (25) we find, by Trigonometry,

$$\tan \theta = \frac{-(A-B) \pm \sqrt{(A-B)^2 + C^2}}{C}$$

Introducing the condition  $4AB = C^2$ , we obtain

$$\begin{aligned} \tan \theta &= -\frac{2A}{C}, \quad \text{if } C \text{ is negative;} \\ &= \frac{2B}{C}, \quad \text{if } C \text{ is positive;} \end{aligned}$$

whence, if  $C$  is negative,

$$\sin \theta = \frac{2A}{\sqrt{4A^2 + C^2}}, \quad \cos \theta = \frac{-C}{\sqrt{4A^2 + C^2}}.$$

And if  $C$  is positive,

If  $C = A = 0$ , the given equation is of the form of (27), its locus is a parabola and can be found as that of (27). If  $C = B = 0$ , the given equation is of the form of (29), and its locus is a parabola.

193. The main results of the investigation are given in the following Table :

LOCI REPRESENTED BY THE GENERAL EQUATION OF THE SECOND DEGREE. $Ax^2 + By^2 + Cxy + Dx + Ey + F = 0.$		
CLASS.	CONDITIONS.	LOCUS.
I. Loci having a centre.	$\Delta$ positive, $\Delta$ not zero. $\Delta$ positive, $\Delta = 0$ . $\Delta$ negative, $\Delta$ not zero. $\Delta$ negative, $\Delta = 0$ .	Ellipse, or no locus. Point. Hyperbola. Two intersecting straight lines.
II. Loci not having a centre.	$\Delta = 0$ , $\Delta$ not zero. $\Delta = 0$ , $\Delta = 0$ .	Parabola. { Two parallel straight lines, One straight line, Or no locus.

Thus it appears that there are no loci of the second order besides those whose properties have been studied in the preceding chapters.

$$\sin \theta = \frac{2B}{\sqrt{4B^2 + C^2}}, \quad \cos \theta = \frac{C}{\sqrt{4B^2 + C^2}}.$$

By substitution we obtain from (26) and (27)

$$U = \frac{2AE - CD}{\sqrt{4A^2 + C^2}}, \quad (30)$$

$$V = \frac{CE - 2BD}{\sqrt{4B^2 + C^2}}, \quad (31)$$

the positive sign being placed before both radicals.

194. EXAMPLES. 1. Determine the nature of the locus

$$5x^2 + 5y^2 + 2xy - 12x - 12y = 0, \quad (1)$$

transform the equation and construct it.

Here  $A=5, B=5, C=2, D=-12, E=-12, F=0$ .

Whence  $\Sigma = 96, \Delta = 1152$ .

Hence the equation represents an ellipse or no real locus.

$$R = \Delta \div \Sigma = 12, h = 1, k = 1.$$

Therefore the equation of the locus referred to new parallel axes through the centre (1, 1) is (§ 188),

$$5x^2 + 2xy + 5y^2 = 12. \quad (2)$$

To cause the term in  $xy$  to disappear, we have

$$\tan 2\theta = \frac{C}{A-B} = \frac{2}{0} = \infty.$$

Whence  $2\theta = 90^\circ$ , or  $\theta = 45^\circ$ .

$$P = \frac{1}{2}[A + B + \sqrt{(A-B)^2 + C^2}] = 6.$$

$$Q = \frac{1}{2}[A + B - \sqrt{(A-B)^2 + C^2}] = 4.$$

(We use the upper signs in the values of  $P$  and  $Q$  by § 189.)

Hence by § 189 the equation of the ellipse referred to its own axes is

$$6x^2 + 4y^2 = 12, \text{ or } \frac{x^2}{2} + \frac{y^2}{3} = 1. \quad (3)$$

Whence  $a = \sqrt{3}, b = \sqrt{2}$ , and  $a$  lies on the axis of  $y$ .

To construct the equation, draw the axes  $O_1 X_1, O_1 Y_1$  (Fig. 75); locate the centre (1, 1). Through this point  $O_2$  draw the second set of axes,  $O_2 X_2, O_2 Y_2$ . Through  $O_2$  draw the third set of axes  $O_2 X_3, O_2 Y_3$ , making  $X_2 O_2 X_3$  equal to  $45^\circ$ .

Lay off  $O_2 A' = O_2 A = \sqrt{3}$ , and  $O_2 O_1 = O_2 B = \sqrt{2}$ . The ellipse having  $BO_1$  and  $AA'$  as axes will be the required locus.



To construct the equation, draw the axes  $O_1X_1$ ,  $O_1Y_1$ , locate the centre  $(-4, 0)$ , and through it draw the third set

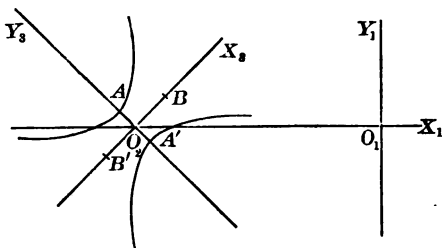


Fig. 76.

of axes  $O_2X_2$ ,  $O_2Y_2$ , making  $X_1O_2X_2 = 45^\circ$ . Then lay off  $O_2A' = O_2A = \sqrt{\frac{2}{3}}$ , and  $O_2B = O_2B' = \sqrt{\frac{2}{3}}$ ; and draw the hyperbola having  $AA'$  and  $BB'$  as its transverse and conjugate axis respectively.

3. Determine the nature of the locus

$$x^2 + 2xy - y^2 + 8x + 4y - 8 = 0, \quad (1)$$

transform the equation and construct it.

Here  $\Sigma = -8$ ,  $\Delta = -176$ ; hence the locus is a hyperbola.

$$R = 22, h = -3, k = -1;$$

hence the first transformed equation is

$$x^2 + 2xy - y^2 = 22. \quad (2)$$

$$\theta = 22\frac{1}{2}^\circ, P = \sqrt{2}, Q = -\sqrt{2};$$

hence the equation of the curve referred to its own axes is

$$\sqrt{2}x^2 - \sqrt{2}y^2 = 22. \quad (3)$$

The hyperbola is equilateral, and its transverse axis lies on the axis of  $x$ . (Let the reader construct it.)

4. Determine the nature of the locus

$$x^2 + y^2 - 2xy + 2x - y - 1 = 0, \quad (1)$$

transform the equation and construct it.



Here  $\Sigma = 0$ ,  $\Delta$  is not zero.

Therefore the locus is a parabola.

$$\theta = 45^\circ, P = 0, Q = 2, U = \frac{1}{2}\sqrt{2}, V = -\frac{3}{2}\sqrt{2};$$

hence, by revolving the axes through an angle of  $45^\circ$ , the equation becomes

$$2y^2 + \frac{1}{2}\sqrt{2}x - \frac{3}{2}\sqrt{2}y - 1 = 0,$$

$$\text{or} \quad y^2 - \frac{3}{4}\sqrt{2}y + \left(\frac{3}{8}\sqrt{2}\right)^2 = -\frac{1}{4}\sqrt{2}x + \frac{25}{8},$$

$$\text{or} \quad \left(y - \frac{3}{8}\sqrt{2}\right)^2 = -\frac{1}{4}\sqrt{2}\left(x - \frac{25}{8}\sqrt{2}\right). \quad (2)$$

Passing to parallel axes whose origin is  $\left(\frac{25}{8}\sqrt{2}, \frac{3}{8}\sqrt{2}\right)$ , (2) becomes  $y^2 = -\frac{1}{4}\sqrt{2}x$ , (3)

the locus of which is a parabola whose latus rectum, or parameter, is  $\frac{1}{4}\sqrt{2}$ .

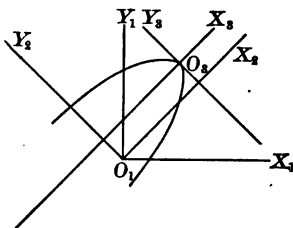


Fig. 77.

To construct the equation, draw the original axes  $O_1X_1, O_1Y_1$ ; then draw the second set of axes  $O_1X_2, O_1Y_2$ , making

$$X_1O_1X_2 = 45^\circ.$$

Locate the new origin  $O_3, \left(\frac{25}{8}\sqrt{2}, \frac{3}{8}\sqrt{2}\right)$ , and through it draw the third set of axes  $O_3X_3, O_3Y_3$ , to which (3) refers the locus, which is now easily drawn.

5. Determine the nature of the locus represented by

$$x^2 - 4xy + 4y^2 - 6x + 12y = 0,$$

and construct it.

Here  $\Sigma = 0$  and  $\Delta = 0$ .

Hence the locus is two parallel straight lines. Factoring the first member of the equation, or solving the equation for  $x$  or  $y$ , we obtain as the equations of these lines

$$y = \frac{1}{2}x, \quad y = \frac{1}{2}x - 3.$$

6. Determine the nature of the locus of

$$y^2 - xy - 6x^2 - 3x + y = 0,$$

and construct it.

Here  $\Sigma$  is negative, and  $\Delta = 0$ .

Hence the locus is two intersecting straight lines. Their equations are

$$y + 2x + 1 = 0, \quad \text{and} \quad y - 3x = 0.$$

195. A **Conic** is the locus of a point whose distance from a fixed point bears a constant ratio to its distance from a fixed straight line.

196. *To find the equation of a conic.*

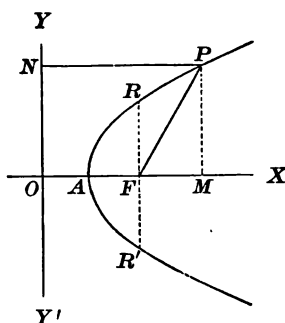


Fig. 78.

Let  $F$  be the fixed point, and  $YY'$  the fixed straight line. Through  $F$  draw  $XO$  perpendicular to  $YY'$ , and use  $OX$  and  $OY$  as axes of reference. Let  $p$  denote the distance  $OF$ , and

$e$  the constant ratio ; then  $FP \div NP = e$ ,  $P$  being any point  $(x, y)$  in the curve.

$$\text{Now} \quad FP^2 = FM^2 + MP^2.$$

$$\text{But} \quad FP = e \times NP = ex,$$

$$FM = x - p, \quad MP = y.$$

$$\text{Therefore} \quad e^2 x^2 = (x - p)^2 + y^2,$$

$$\text{or} \quad (1 - e^2)x^2 + y^2 - 2px + p^2 = 0, \quad (1)$$

which is the equation required.

COR. In equation (1), which is of the second degree,

$$\Sigma = 4(1 - e^2), \text{ and } \Delta = 4p^2e^2.$$

Hence, when the fixed point is without the fixed line,  $\Delta$  is not zero, and

If  $e < 1$ ,  $\Sigma > 0$ , and the conic is an ellipse.

If  $e = 1$ ,  $\Sigma = 0$ , and the conic is a parabola.

If  $e > 1$ ,  $\Sigma < 0$ , and the conic is an hyperbola.

When the fixed point is in the fixed line,  $\Delta = 0$ , and

If  $e < 1$ ,  $\Sigma > 0$ , and the conic is the point  $(0, 0)$ .

If  $e = 1$ ,  $\Sigma = 0$ , and the conic is a right line.

If  $e > 1$ ,  $\Sigma < 0$ , and the conic is two intersecting right lines.

If  $e = 0$ , by § 61, the conic is a circle or a point, according as  $p$  is not or is zero.

From §§ 92, 150, 183, it follows that the fixed point is a **Focus**, the fixed right line a **Directrix**, and the constant ratio the **Eccentricity**, of the conic.

#### EX. 40.

Determine the nature of the following loci, transform their equations, and construct them.

$$1. \quad 3x^2 + 2y^2 - 2x + y - 1 = 0.$$

$$2. \quad 3x^2 + 2xy + 3y^2 - 16y + 23 = 0.$$

3.  $x^2 - 10xy + y^2 + x + y + 1 = 0.$
4.  $x^2 + xy + y^2 + x + y - 5 = 0.$
5.  $y^2 - x^2 - y = 0.$
6.  $1 + 2x + 3y^2 = 0.$
7.  $y^2 - 2xy + x^2 - 8x + 16 = 0.$
8.  $x^2 - 2xy + y^2 - 6x - 6y + 9 = 0.$
9.  $y^2 - 2x - 8y + 10 = 0.$
10.  $4x^2 + 9y^2 + 8x + 36y + 4 = 0.$
11.  $52x^2 + 72xy + 73y^2 = 0.$
12.  $9y^2 - 4x^2 - 8x + 18y + 41 = 0.$
13.  $y^2 - xy - 5x + 5y = 0.$

## CHAPTER IX.

### HIGHER PLANE CURVES.

197. An **Algebraic Curve** is one whose rectilinear equation contains only algebraic functions. A **Transcendental Curve** is one whose rectilinear equation involves other than algebraic functions. Thus the loci of  $y = a^x$ ,  $y = \tan x$ ,  $y = (a - x) \tan(\frac{1}{2} \pi x \div a)$ ,  $y = \sin^{-1} x$  are transcendental curves. Transcendental curves, and all algebraic curves above the second order, are called **Higher Plane Curves**.

Let the symbol  $F(x, y)$  denote any rational integral function of  $x$  and  $y$ , of the third or higher degree. If  $F(x, y)$  breaks up into simple or quadratic factors in  $x$  and  $y$ , the locus of  $F(x, y) = 0$  consists of lines of the first or second order. If  $F(x, y)$  does not break up into rational factors in  $x$  and  $y$ , the locus of  $F(x, y) = 0$  is a higher plane curve whose order is of the degree of the equation. Thus the locus of

$$y^3 - x^3 = (y - x)(y^2 + xy + x^2) = 0,$$

consists of the right line  $y - x = 0$  and the ellipse  $y^2 + xy + x^2 = 0$ ; the locus of

$$x^4 + x^2y^2 + 2y^3 - 2x^4 - 5x^2 - 3 = (y^2 + 2x^2 + 3)(y^2 - x^2 - 1) = 0,$$

consists of the ellipse  $y^2 + 2x^2 + 3 = 0$ , and the hyperbola  $y^2 - x^2 - 1 = 0$ ; while the locus of  $y^3 - ax + x^3 = 0$  is a higher plane curve of the third order.

In this chapter we shall consider a few of the higher plane curves, some of which possess historic interest from the labor bestowed on them by the ancient mathematicians.

198. The **Cissoid of Diocles**. Let  $XH$  be a tangent to the circle  $XSO$  at the vertex of any diameter  $OX$ ; let  $OR$  be

any right line from  $O$  to  $XH$  cutting the circle at  $S$ , and take  $OP = RS$ ; then the locus of  $P$ , as  $OR$  revolves about  $O$ , is the *Cissoid*.

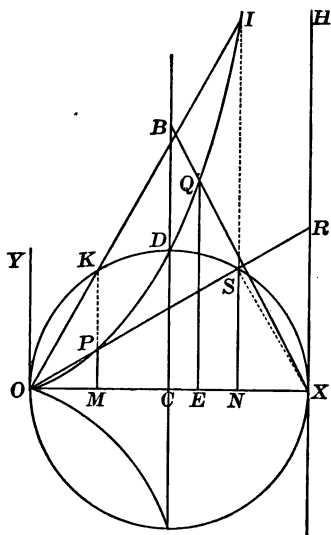


Fig. 79.

To find its equation referred to the rectangular axes  $OX$  and  $OY$ , let  $OM = x$ ,  $MP = y$ , and  $OC = CX = CD = r$ .

$$\text{Now} \quad MP : OM :: NS : ON. \quad (1)$$

$$\text{Since} \quad OP = RS, \quad OM = NX.$$

$$\begin{aligned} \text{Hence} \quad ON &= OX - NX = OX - OM = 2r - x, \\ \text{and} \quad NS &= \sqrt{ON \times NX} = \sqrt{(2r - x)x}. \end{aligned}$$

Substituting these values in (1), we obtain

$$\frac{y}{x} = \frac{\sqrt{(2r - x)x}}{2r - x}, \quad \text{or} \quad y^2 = \frac{x^3}{2r - x}, \quad [48]$$

which is the equation sought.

A discussion of [48] leads to the following conclusions:

- (i.) The curve lies between the lines  $x = 0$  and  $x = 2r$ .
- (ii.) It is symmetrical with respect to the axis of  $x$ .
- (iii.) It passes through the extremities of the diameter perpendicular to  $OX$ .
- (iv.) It has two infinite branches to which  $x = 2r$  is an asymptote.

COR. 1. To find the polar equation let  $\theta = XOP$  and  $\rho = OP$ ;

then  $\cos \theta = \frac{OX}{OR} \text{ or } \frac{OS}{OX}$

Hence  $\rho = OP = SR = OR - OS = \frac{2r}{\cos \theta} - 2r \cos \theta$   
 $= 2r \frac{\sin^2 \theta}{\cos \theta}.$

Therefore  $\rho = 2r \sin \theta \tan \theta$ ,  
 which is the equation sought.

The cissoid was invented by Diocles, a Greek mathematician of the sixth century A.D, for the purpose of constructing the solution of the problem of *finding two mean proportionals*, of which the *duplication of the cube* is a particular case.

COR. 2. To duplicate the cube, in Fig. 79, take  $CB = 2r$ , and draw  $BX$  cutting the cissoid in  $Q$ ; then, since  $CX = \frac{1}{2}CB$ ,  $EX = \frac{1}{2}EQ$ . But from the equation of the cissoid, we have

$$EQ^3 = \frac{OE^3}{EX} = \frac{OE^3}{\frac{1}{2}EQ}; \text{ therefore } EQ^3 = 2OE^3.$$

Now let  $c$  denote the edge of any given cube; take  $c_1$  so that

$$OE : EQ :: c : c_1, \text{ or } OE^3 : EQ^3 :: c^3 : c_1^3.$$

But  $EQ^3 = 2OE^3$ , therefore  $c_1^3 = 2c^3$ ;

that is,  $c_1$  is the edge of a cube double the given cube in volume. In like manner, by taking  $CB = mr$ , we can find the edge of a cube  $m$  times the given cube in volume.

**199. The Conchoid of Nicomedes.** The Conchoid is the locus of a point  $P$  such that its distance from a fixed line  $XX'$ , measured along the line through  $P$  and a fixed point  $A$ , is constant.  $A$  is the Pole,  $XX'$  the Directrix, and the constant distance  $RP$ , denoted by  $b$ , is the Parameter.

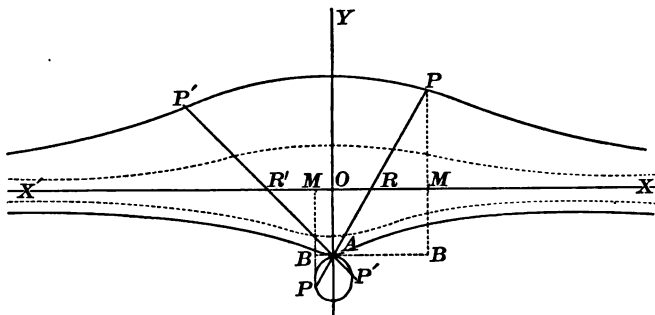


Fig. 80.

To construct the conchoid by points, through  $A$  draw any line  $AP$  cutting  $XX'$  in  $R$ . Lay off  $RP = b$  on both sides of  $XX'$ . In like manner locate the points  $P'$ , and any number of others, and trace the branches through them.

To find the equation of the curve referred to  $XX'$  and a line through  $A$  perpendicular to  $XX'$ ; let

$$OM = x, MP = y, AO = a.$$

Now  $AB : BP :: RM : MP,$

and  $RM = \sqrt{RP^2 - MP^2} = \sqrt{b^2 - y^2}.$

Therefore  $x : y + a :: \sqrt{b^2 - y^2} : y.$

Therefore  $x^2 y^2 = (y + a)^2 (b^2 - y^2),$  [49]

which is the equation of both branches of the conchoid. A discussion of [49] leads to the following conclusions:

- (i.) The curve lies between the lines  $y = b$  and  $y = -b$ .
- (ii.) The curve is symmetrical with respect to the axis of  $y$ .



(iii.) The axis of  $x$  is an asymptote to each branch of the curve.

If  $b > a$ , the lower branch has an oval or loop, as in the figure.

If  $b = a$ , the lower branch passes through  $A$  and is somewhat like that in the figure, without the loop below  $A$ .

If  $b < a$ , the upper and lower branches are like the dotted lines in the figure.

If  $a = 0$ , the conchoid becomes a circle.

COR. 1. If  $A$  be the pole and  $AY$  the polar axis, we have

$$\rho = AR' \pm R'P' = a \sec \theta \pm b,$$

which is the polar equation of the conchoid.

The conchoid, invented by Nicomedes, a Greek mathematician of the second century A.D., was, like the cissoid, first formed for solving the problem of *finding two mean proportionals* or *duplicating the cube*. It is more readily applicable, however, to the *trisection of an angle*, a problem not less celebrated among the ancients.

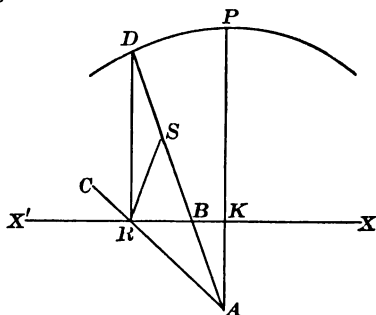


Fig. 81.

COR. 2. To trisect any angle, as  $CAP$  (Fig. 81), on  $AC$  lay off  $AR$  any length; through  $R$  draw  $RK$  perpendicular to  $AP$ , and take  $KP = 2RA$ . Construct a conchoid with  $A$  as a pole,  $XX'$  as a directrix, and  $KP$  as a parameter.

At  $R$  erect a perpendicular to  $XX'$  intersecting the conchoid in  $D$ ; then  $DA$  will trisect the angle  $RAP$ .

For bisect  $DB$  in  $S$ ; then

$$RS = SD = \frac{1}{2} KP = RA.$$

Therefore  $\angle RAS = \angle RSA = 2\angle RDS = 2\angle DAP$ .

Therefore  $\angle DAP = \frac{1}{2}\angle RAP$ .

**200. The Lemniscate of Bernoulli.** The *Lemniscate* is the locus of the intersection of a tangent to a rectangular hyperbola with the perpendicular to it from the centre.

To find its equation we proceed as follows:

The equation of the tangent to the equilateral hyperbola  $x^2 - y^2 = a^2$  at the point  $(x_1, y_1)$  is

$$x_1x - y_1y = a^2 \quad (1)$$

The equation of the perpendicular from the origin to (1) is

$$y = -\frac{y_1}{x_1}x, \text{ or } \frac{x}{x_1} = -\frac{y}{y_1}. \quad (2)$$

Solving (1) and (2) for  $x_1$  and  $y_1$  by multiplying each term of (1) by one of the members of (2), we obtain

$$x^2 + y^2 = \frac{a^2x}{x_1} = -\frac{a^2y}{y_1}.$$

$$\text{Therefore } x_1 = \frac{a^2x}{x^2 + y^2}, \quad y_1 = -\frac{a^2y}{x^2 + y^2}.$$

But, since  $(x_1, y_1)$  is on the hyperbola, we have  $x_1^2 - y_1^2 = a^2$ ; hence, by substitution, we obtain

$$(x^2 + y^2)^2 = a^2(x^2 - y^2), \quad [50]$$

which is the equation sought.

From [50] it follows that the curve is symmetrical with respect to both axes. The form of the curve is given in the figure.

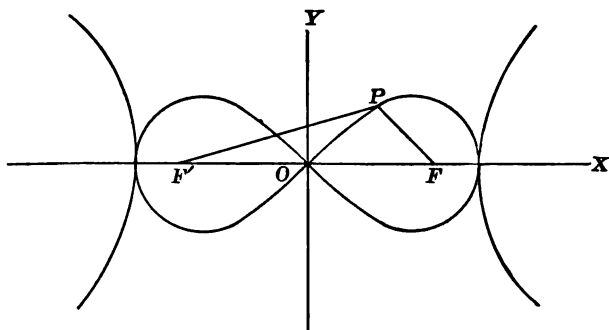


Fig. 82.

COR. 1. Substituting  $\rho \cos \theta$  for  $x$ , and  $\rho \sin \theta$  for  $y$  in [50], and remembering that  $\cos^2 \theta - \sin^2 \theta = \cos 2\theta$ , we obtain

$$\rho^2 = a^2 \cos 2\theta \quad (2)$$

as the polar equation of the lemniscate.

COR. 2. From (2),  $\rho = \pm a \sqrt{\cos 2\theta}$ .

Hence, when  $\theta = 0$ ,  $\rho = \pm a$ . As  $\theta$  increases from  $0^\circ$  to  $45^\circ$ ,  $\rho$  changes from  $\pm a$  to  $\pm 0$ , and the portions in the first and third quadrants are traced. As  $\theta$  increases from  $45^\circ$  to  $135^\circ$ ,  $\cos 2\theta$  is negative and  $\rho$  is imaginary. As  $\theta$  increases from  $135^\circ$  to  $180^\circ$ ,  $\rho$  changes from  $\pm 0$  to  $\pm a$ , and the portions in the second and fourth quadrants are traced. Therefore

- (i.) The curve consists of two ovals meeting at the pole  $O$ .
- (ii.) The tangents to the curve at  $O$  are the asymptotes to the equilateral hyperbola.

201. The Witch of Agnesi. Let  $YH$  be a tangent to the circle  $OKY$  at the vertex of the diameter  $OY$ ; let  $OR$  be any

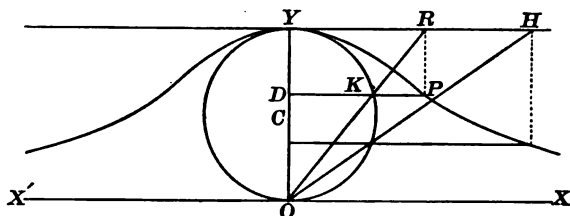


Fig. 83.

line from  $O$  to  $YH$  cutting the circle in  $K$ ; produce the ordinate  $DK$ , and make  $DP = YR$ ; then the locus of  $P$  is the *Witch*.

To find its equation, let the tangent  $OX$  and the diameter  $OY$  be the axes; let  $OY = 2r$  and  $P$  be any point; then  $OD = y$ ,  $DP = x$ , and

$$OD : OY :: DK : DP (= YR), \quad (1)$$

or  $y : 2r :: \sqrt{y(2r - y)} : x.$

Therefore  $x^2 y = 4r^2(2r - y)$  [51]

is the equation of the *Witch*.

COR. 1. Since  $x = \pm 2r \sqrt{(2r - y) \div y}$ , it follows that

- (i.) The curve is symmetrical with respect to the axis of  $y$ .
- (ii.) The curve lies between the lines  $y = 0$  and  $y = 2r$ .
- (iii.) The axis of  $x$  is an asymptote to each infinite branch.

COR. 2. From (1) it follows that corresponding abscissas of the circle and the *Witch* are proportional to the ordinate and the diameter of the circle.

The *Witch* was invented by Donna Maria Agnesi, an Italian mathematician of the eighteenth century.

**202. The Cycloid.** A Cycloid is generated by a point  $P$  in the circumference of a circle  $RPC$ , which rolls along a right line  $OX$ . The curve consists of an unlimited number of branches, but a single branch is usually termed a *cycloid*. The right line  $OX$  is called the **Base**; the rolling circle  $RPC$  the **Generatrix**; and  $P$  the **Generating Point**. If  $OK = KX$ , the perpendicular  $KH$  is the **Axis**, and  $H$  the **Highest Point**.

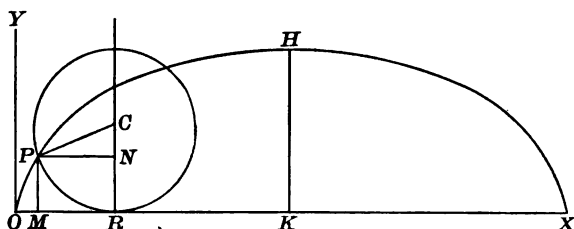


Fig. 84.

To find the equation of the curve, first take as the axis of  $X$  the base  $OX$ , and as the origin the point  $O$ , where the curve meets the base. Let  $r$  denote the radius of the generatrix  $RPC$ , and  $\theta$  the angle  $PCR$ ; then arc  $PR$  equals  $OR$  over which it has rolled, and  $\theta = \text{arc } PR \div r$ . Denote the co-ordinates of  $P$  by  $x$  and  $y$ ; then

$$x = OM = OR - MR = \text{arc } PR - PN = r\theta - r \sin \theta.$$

$$y = MP = RC - NC = r - r \cos \theta.$$

Therefore

$$\left. \begin{aligned} x &= r(\theta - \sin \theta) \\ y &= r(1 - \cos \theta) \end{aligned} \right\} \quad (1)$$

Equations (1) taken as simultaneous are the equations of the cycloid.

To eliminate  $\theta$  between these equations, from the second, we obtain

$$\cos \theta = \frac{r-y}{r}; \text{ therefore, } \sin \theta = \pm \frac{\sqrt{2ry-y^2}}{r};$$

$$\text{and } \text{vers } \theta = [1 - \cos \theta] = \frac{y}{r}; \text{ or } \theta = \text{vers}^{-1} \frac{y}{r}.$$

Substituting these values of  $\theta$  and  $\sin \theta$  in the first of equations (1), we have

$$x = r \text{vers}^{-1} \frac{y}{r} \mp \sqrt{2ry-y^2}, \quad [52]$$

which is the equation of the cycloid in the more common form.

In the value of  $\sin \theta$ , and therefore in equation [52], the upper or lower sign is used according as  $\theta < \text{or } > \pi$ ; that is, according as the point is on the first or second half of a branch.

From  $y = r(1 - \cos \theta)$  it follows that the locus lies between the lines  $y = 0$  and  $y = 2r$ .

For  $y = 0$  in [52],  $x = 0, \pm 2\pi r, \pm 4\pi r, \dots$ ; hence, the locus consists of an unlimited number of branches like  $OHX$ , both to the right and left of  $OY$ .

**203.** Let the highest point  $O$  (Fig. 85) be taken as the origin, and  $OX$  parallel to the base as the axis of  $x$ ; then  $OY$ , the axis of the curve, will be the axis of  $y$ . Let  $\theta$  denote the angle  $HCK$ .

The point  $K$  was at  $Y$  when  $P$  was at  $O$ , and arc  $KH = YH$ .

Hence  $x = OM = YH + BP = r\theta + r \sin \theta$ .

$$y = -MP = -NC + BC = -r + r \cos \theta.$$

Hence the equations are

$$\left. \begin{aligned} x &= r(\theta + \sin \theta) \\ y &= r(\cos \theta - 1) \end{aligned} \right\}, \quad (1)$$

$$\text{or } x = r \text{vers}^{-1} \left( \frac{-y}{r} \right) \pm \sqrt{-2ry-y^2}. \quad (2)$$

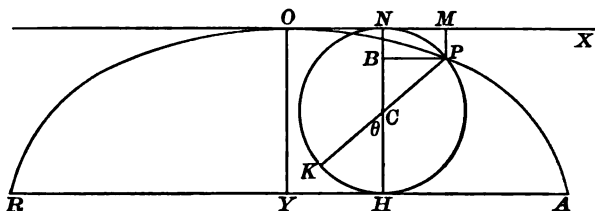


Fig. 85.

The invention of the cycloid is usually ascribed to Galileo. After the conics, no curve has exercised the ingenuity of mathematicians more than the cycloid, and their labors have been rewarded by the discovery of a multitude of interesting properties. For example, the length of the branch  $ROA$  is eight times the radius, and the area  $ROA$  is three times the area of the generating circle.

**Ex. 41.**

1. Prove that the cissoid is the locus of the intersection of a tangent to the parabola  $y^2 = -8rx$  with the perpendicular to it from the origin.

By § 112, the equation of the tangent to  $y^2 = -8rx$  is

$$y = mx - \frac{2r}{m}, \quad (1)$$

and the perpendicular to it from  $(0, 0)$  is

$$y = -\frac{1}{m}x; \text{ therefore, } m = -\frac{x}{y}. \quad (2)$$

Eliminating  $m$  between (1) and (2), we obtain

$$y^2 = \frac{x^3}{2r - x}.$$

2. At the centre of any circle  $C$  (Fig. 86), erect  $CH \perp$  to the diameter  $OX$ ; and on  $XO$  produced lay off  $OA = OC = r$ . Let  $LQR$  be a rectangular ruler of which the leg  $QR$  equals  $2r$ . If the ruler be moved so that the leg  $LQ$  slides along  $A$ ,





3. In Fig. 79 prove that  $NS$  and  $ON$  are two mean proportionals between  $OM$  and  $NI$ ; that is, prove

$$OM : NS : ON : NI.$$

$$NS^2 = NX \times ON = OM \times ON.$$

The right line  $OI$  will pass through  $K$ ; hence

$$\begin{aligned}\angle OIN &= \angle YOI = \frac{1}{2} \text{arc } OK = \frac{1}{2} \text{arc } SX \\ &= \angle SOX;\end{aligned}$$

therefore  $NS : ON :: ON : NI$ , etc.

4. If in the lemniscate (Fig. 82),  $OF' = OF = \frac{1}{2}a\sqrt{2}$ , prove that  $FP \times F'P$  is constant,  $P$  being any point on the curve; and hence that the lemniscate may be defined as the locus of a point the product of whose distances from two fixed points is constant.

5. Construct the logarithmic curve  $y = a^x$ , or  $x = \log_a y$ . Prove that every logarithmic curve passes through the point  $(0, 1)$ , and has the axis of  $x$  as an asymptote.

6. The **Trochoid** is the curve traced by any point in the radius of a circle rolling on a right line. If  $r$  denote the radius of the circle,  $b$  the distance of the generating point from its centre, and  $\phi$  denote the same angle as in § 202, show that the equations of the trochoid are

$$\left. \begin{aligned}x &= r\phi - b \sin \phi \\ y &= r - b \cos \phi\end{aligned} \right\}. \quad (1)$$

When  $b < r$ , the trochoid is called the **Prolate Cycloid**; and when  $b > r$ , the **Curtate Cycloid**. When  $b = r$ , the curve is the *cycloid*.

#### SPIRALS.

**204.** A **Spiral** is the locus of a point whose radius vector continually increases, or continually decreases, while its vectorial angle increases (or decreases) without limit.

A **Spire** is the portion of the spiral traced during one revolution of the radius vector.

The **Measuring Circle** is the circle whose centre is the pole and whose radius is the value of  $\rho$  when  $\theta = 2\pi$ .

**205. The Spiral of Archimedes.** If the radius vector of a moving point has a constant ratio to its vectorial angle; that is, if

$$\rho = c\theta, \quad (1)$$

the locus is the Spiral of Archimedes.

From equation (1), we have

When  $\theta = 0, \frac{1}{4}\pi, \frac{2}{4}\pi, \frac{3}{4}\pi, \pi, \frac{5}{4}\pi, \frac{6}{4}\pi, \frac{7}{4}\pi, 2\pi, \frac{9}{4}\pi, \dots$

$\rho = 0, \frac{1}{4}\pi c, \frac{2}{4}\pi c, \frac{3}{4}\pi c, \pi c, \frac{5}{4}\pi c, \frac{6}{4}\pi c, \frac{7}{4}\pi c, 2\pi c, \frac{9}{4}\pi c, \dots$

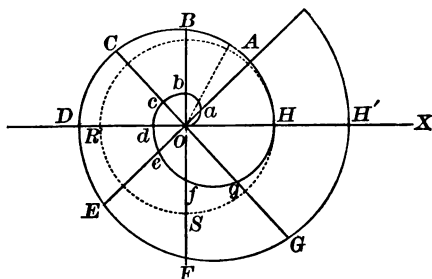


Fig. 87.

Hence to construct the Spiral of Archimedes, draw the radial lines  $OH', OA, OB, \dots OG$ , including angles of  $\frac{1}{4}\pi$ ; on these lay off  $Oa = \frac{1}{4}\pi c, Ob = \frac{2}{4}\pi c, Oc = \frac{3}{4}\pi c, \dots, OH = 2\pi c$ , and trace the first spire  $OabcdefgH$  through these points. Any number of other spires are easily constructed by noting that the distance between two spires, measured on a radius vector, is equal to  $2\pi c$ . Thus, by taking  $aA, bB, cC, dD, eE, fF, gG, HH'$ , each equal to  $2\pi c$ , we obtain points of the second spire; and so on. Any number of additional radial lines may be drawn to locate points in the curve,

The spires, being everywhere equally distant along radial lines, are said to be parallel. The measuring circle is  $HRH$ , whose radius is  $OH$  or  $2\pi c$ .

**206. The Reciprocal or Hyperbolic Spiral.** If the radius vector of a point varies inversely as its vectorial angle; that is, if

$$\rho\theta = c, \quad (1)$$

the locus is the **Reciprocal Spiral**.

Since  $\rho = c \div \theta$ , we have

when  $\theta = 0, \frac{1}{4}\pi, \frac{1}{2}\pi, \frac{3}{4}\pi, \pi, \frac{5}{4}\pi, \frac{3}{2}\pi, \frac{7}{4}\pi, 2\pi, \dots$

$$\rho = \infty, 8\frac{c}{2\pi}, 4\frac{c}{2\pi}, 8\frac{c}{2\pi}, 2\frac{c}{2\pi}, 8\frac{c}{2\pi}, 4\frac{c}{2\pi}, 8\frac{c}{2\pi}, \frac{c}{2\pi}, \dots$$

Hence the radius of the measuring circle is  $c \div 2\pi$ , and its circumference is  $c$ .

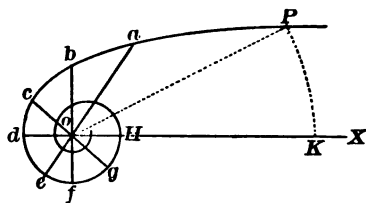


Fig. 88.

To construct the curve draw the radial lines  $OX, Oa, Ob, Oc, Od, Oe, Of, Og$ , including angles of  $\frac{1}{4}\pi$ . Take  $OH = c \div 2\pi$ , and lay off  $Oa = 8 \times OH, Ob = 4 \times OH, Oc = \frac{8}{3} \times OH, Od = 2 \times OH$ , etc.; and through the points  $a, b, c, d$ , etc., trace the curve. In like manner, any number of spires may be drawn. From (1) it is evident that  $\rho$  approaches zero, as  $\theta$  approaches infinity; that is, the curve continually approaches the pole without ever reaching it.

Since  $\rho\theta = c$ , it follows that the arc  $PK$  described with the radius vector of any point  $P$  is constant and equal to  $c$ .

Now, as  $\rho$  approaches infinity, this arc approaches a perpendicular to  $OX$ . Hence the line parallel to  $OX$  at the distance of  $c$  above it is an asymptote to the infinite branch of the spiral.

**207. The Lituus.** If the square of the radius vector of a point varies inversely as its vectorial angle; that is, if

$$\rho^2\theta = c,$$

the locus is the **Lituus**.

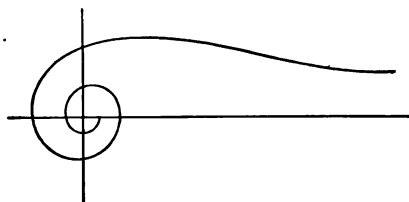


Fig. 89.

Let the student construct the curve from its equation and show that

(i.) The curve continually approaches the pole without ever reaching it, as  $\theta$  increases without limit.

(ii.) The polar axis is an asymptote to the infinite branch.

**208. The Logarithmic Spiral.** If the radius vector of a point increases in a geometrical ratio, while its vectorial angle increases in an arithmetical ratio; that is, if

$$\rho = a^\theta, \text{ or } \theta = \log_a \rho, \quad (1)$$

the locus is the **Logarithmic Spiral**.

Since  $\rho = 1$  when  $\theta = 0$ , every logarithmic spiral passes through the point  $(1, 0)$ .

To construct a logarithmic spiral let  $a = 2$ ; then  $\rho = 2^\theta$ .

When  $\theta = 0$ ,  $1 (= 57^\circ.3)$ ,  $2 (= 114^\circ.6)$ , .....,  $2\pi$ .

$\rho = 1$ ,  $2$ ,  $4$ , .....,  $77.88$ .

In Fig. 90 let  $XOb = 57.3^\circ$ ,  $XOc = 114.6^\circ$ ,  $Oa = 1$ ,  $Ob = 2$ ,  $Oc = 4$ , then  $a, b, c$  are three points on the spiral. As  $\theta$  increases,  $\rho$  increases rapidly, but it becomes infinity only when  $\theta$  does; and hence only after an infinite number of revolu-

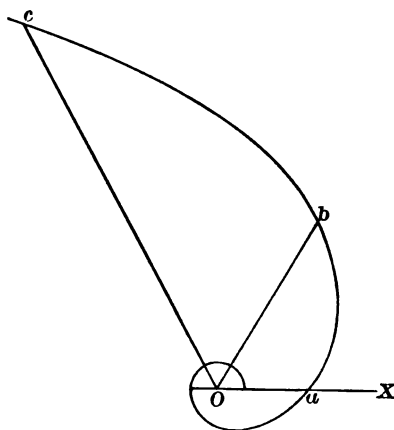


Fig. 90.

tions. As  $\theta$  decreases from zero,  $\rho$  decreases from unity. Since  $\rho$  approaches zero as  $\theta$  approaches negative infinity, the curve approaches the pole without ever reaching it.

**209. The Parabolic Spiral.** If in the equation  $y^2 = 4px$ , the values of  $x$  be laid off from  $A$  (Fig. 91) on the circle  $AH$ , and those of  $y$  on its corresponding radii produced, the locus of the point thus determined is the **Parabolic Spiral**.

To find its equation, denote the radius  $OA$  by  $r$ ; and let  $P$  be any point, then

$$x = AmH = r\theta.$$

$$y = HP = OP - OH = \rho - r,$$

Substituting these values of  $x$  and  $y$  in  $y^2 = 4px$ , we obtain the polar equation

$$(\rho - r)^2 = 4pr\theta.$$

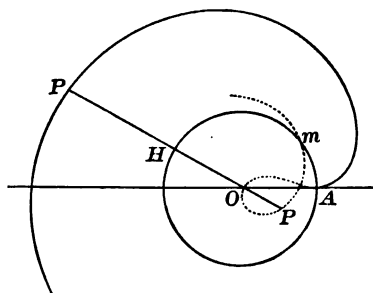


Fig. 91.

The curve consists of two branches beginning at  $A$ ; the one determined by the positive values of  $y$  is an infinite spiral lying entirely without the circle; the other branch passes through the pole, forms a loop, and passes without the circle when  $\rho = -r$ , and  $\theta = r \div p$ .

NOTE. Among the ancients no problems were more celebrated than the "Duplication of the Cube" and the "Trisection of an Angle." Hippocrates of Chios reduced these two problems to the more general problem of *finding two mean proportionals between two given lines*. Thus, if  $c$  be the edge of the given cube, and  $x$  and  $y$  be the two mean proportionals between  $c$  and  $2c$ , we have

$$c : x = x : y = y : 2c.$$

Therefore

$$\left(\frac{c}{x}\right)^3 = \frac{c}{x} \times \frac{x}{y} \times \frac{y}{2c} = \frac{1}{2}, \text{ or } x^3 = 2c^3.$$

Hence  $x$ , the first of the two mean proportionals between  $c$  and  $2c$ , is the edge of a cube double the given cube in volume.

After years of vain endeavor to solve these problems by the right line and the circle, the ancient geometers began to invent and study other curves, as the conics and some of the higher plane curves. The invention of the conics is credited to Plato, in whose school their properties were an object of special study.

## PART II.—SOLID GEOMETRY.

### CHAPTER I.

#### THE POINT.

**210.** The position of a point in space may be determined by referring it to three fixed planes meeting in a point. The fixed planes are called **Co-ordinate Planes**, their lines of intersection the **Co-ordinate Axes**, and their common point the **Origin**. In what follows we shall employ co-ordinate planes that intersect each other at right angles.

Let  $XOY$ ,  $YOZ$ ,  $ZOX$ , be three planes of indefinite extent intersecting each other at right angles in the lines  $XX'$ ,  $YY'$ ,

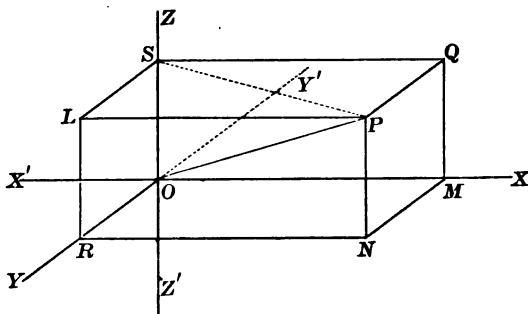


Fig. 92.

$ZZ'$ . These co-ordinate planes are called the planes  $xy$ ,  $yz$ ,  $zx$ , respectively; the axes  $XX'$ ,  $YY'$ ,  $ZZ'$ , are called the axes of  $x$ ,  $y$ ,  $z$ , respectively; and their common point  $O$  is the origin.

The co-ordinate planes divide all space into eight portions, called **Octants**, which are numbered as follows: The *First Octant* is  $O-XYZ$ , the *Second*  $O-YX'Z$ , the *Third*  $O-X'Y'Z$ , the *Fourth*  $O-Y'XZ$ , the *Fifth*  $O-XYZ'$ , the *Sixth*,  $O-YX'Z'$ , the *Seventh*  $O-X'Y'Z'$ , the *Eighth*  $O-Y'XZ'$ . The fifth octant is below the first.

Let  $P$  be any point in space, and through it pass three planes parallel respectively to the three co-ordinate planes, thus forming the rectangular parallelopiped  $P-ORNM$ . The position of  $P$  will be determined when we know the lengths and directions of the lines  $LP$ ,  $QP$ ,  $NP$ . These three lines are called the **Rectangular Co-ordinates**  $x$ ,  $y$ ,  $z$ , of the point  $P$ , which is written  $(x, y, z)$ .

A co-ordinate is *positive* when it has the direction of  $OX$ ,  $OY$ , or  $OZ$ ; hence it is *negative* when it has the direction of  $OX'$ ,  $OY'$ , or  $OZ'$ .

Thus the co-ordinate  $x$  is positive or negative according as it extends to the right or to the left from the plane  $yz$ ;  $y$  is positive or negative according as it extends to the front or to the rear of the plane  $zx$ ; and  $z$ , according as it extends upward or downward from the plane  $xy$ . Hence the octant in which a point is situated is determined by the *signs* of its co-ordinates. Since the first octant has the positive directions of the axes for its edges, the co-ordinates of a point in the first octant are all positive. If  $(a, b, c)$  be a point in the first octant, the corresponding point

in the second octant is  $(-a, b, c)$ ,  
 in the third octant is  $(-a, -b, c)$ ,  
 in the fourth octant is  $(a, -b, c)$ ,  
 in the fifth octant is  $(a, b, -c)$ ,  
 in the sixth octant is  $(-a, b, -c)$ ,  
 in the seventh octant is  $(-a, -b, -c)$ ,  
 in the eighth octant is  $(a, -b, -c)$ .



The point  $(x, y, 0)$  is in the plane  $xy$ .

The point  $(x, 0, 0)$  is in the axis of  $x$ .

The point  $(0, 0, 0)$  is the origin.

The lines  $OM$ ,  $OR$ ,  $OS$ , or  $OM$ ,  $MN$ ,  $NP$ , may be taken as the co-ordinates of  $P$ , for they have the same length and direction as  $LP$ ,  $QP$ ,  $NP$ , respectively. To construct  $P$ ,  $(x, y, z)$ , we take  $OM = x$ , draw  $MN$  parallel to  $OY$ , take  $MN = y$ , draw  $NP$  parallel to  $OZ$ , and take  $NP = z$ .

**211.** The **Radius Vector** of a point is the line drawn to it from the origin. Thus  $OP (= \rho)$  is the radius vector of  $P$ . From the rectangular parallelopiped in Fig. 92, we have

$$OP^2 = OM^2 + MN^2 + NP^2.$$

Hence, denoting the co-ordinates of  $P$  by  $x, y, z$ , we have

$$\rho^2 = x^2 + y^2 + z^2. \quad [53]$$

*That is, the square of the radius vector of a point is equal to the sum of the squares of its rectangular co-ordinates.*

**212.** By the angle between two non-intersecting straight lines is meant the angle between any two intersecting straight lines that are parallel to them. Thus, any line parallel to  $OP$  (Fig. 92) makes the angles  $XOP$ ,  $YOP$ ,  $ZOP$  with the axes of  $x, y, z$ , respectively.

The angles which a line makes with the positive directions of the co-ordinate axes are called its **Direction Angles**; and the cosines of these angles are called the **Direction Cosines** of the line.

The direction angles of a line are always positive and cannot exceed  $\pi$ , or  $180^\circ$ .

**213.** Let  $\alpha, \beta, \gamma$  denote the direction angles of  $OP$  (Fig. 92), or any line parallel to it, and  $x, y, z$  the co-ordinates of  $P$ , and let  $\rho = OP$ ; then evidently

$$x = \rho \cos \alpha, y = \rho \cos \beta, z = \rho \cos \gamma. \quad [54]$$

Squaring and adding [54], and substituting in [53], we obtain

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1. \quad [55]$$

That is, *the sum of the squares of the direction cosines of a line is equal to unity.*

COR. Whatever be the values of  $x, y, z$  in [54], if they each be divided by  $\rho$ , or  $\sqrt{x^2 + y^2 + z^2}$ , the quotients are the direction cosines of the radius vector of the point  $(x, y, z)$ .

Hence, *if any three real quantities be each divided by the square root of the sum of their squares, the quotients are the direction cosines of some line.*

#### Ex. 42.

1. In what octants may  $(x, y, z)$  be, when  $x$  is positive? when  $x$  is negative? when  $y$  is positive? when  $y$  is negative? when  $z$  is positive? when  $z$  is negative?

2. In what octant is  $(-2, 4, 6)$ ?  $(2, 4, -3)$ ?  $(-2, 4, -1)$ ?  $(-2, -3, -1)$ ?  $(-2, -3, 3)$ ?  $(2, -3, 1)$ ?  $(2, -1, -3)$ ? Construct each point.

3. In what line is  $(a, 0, 0)$ ?  $(0, 0, c)$ ?  $(0, b, 0)$ ?

4. In what plane is  $(a, b, 0)$ ?  $(a, 0, c)$ ?  $(0, b, c)$ ?

5. Find the length of the radius vector of  $(3, 4, 5)$ ,  $(2, -3, -1)$ ,  $(7, -3, -5)$ . Find the direction cosines of the radius vector of each point.

6. The direction cosines of a line are proportional to 1, 2, 3; find their values. What is the direction of the line?

7. What is the direction of the line whose direction cosines are proportional to  $A, B, C$ ? What are the values of its direction cosines?

8. Two direction angles of a line are  $60^\circ$  and  $45^\circ$ , what is the third? If two are  $60^\circ$  and  $30^\circ$ , what is the third? If two are  $135^\circ$  and  $60^\circ$ , what is the third?

**214. Projections.** The *projection* of a point upon a right line is the foot of the perpendicular from the point to the line; or it is the intersection of the line with the plane through the point perpendicular to the line. Thus  $M, R, S$  (Fig. 92) are the projections of the point  $P$  upon the axis of  $x, y, z$ , respectively. Here and in the following pages, by projection is meant the orthogonal projection.

The *projection* of a limited right line on another right line is the part intercepted between the projections of its extremities.

Hence,  $OM, OR, OS$  (Fig. 92), are the projections of  $OP$  on the axes of  $x, y, z$ , respectively. That is, *the co-ordinates of any point are the projections of its radius vector on the three axes.*

**215.** The projections of any line  $PQ$  on parallel lines are equal; for these projections are parallel lines included between parallel planes through  $P$  and  $Q$ . Now the projection of any straight line on another that passes through one of its extremities is evidently equal to the product of its length into the cosine of their included angle.

Hence, *the projection of any limited straight line on any other straight line is equal to its length multiplied by the cosine of the angle between the lines.*

**216.** Let  $AD$  (Figs. 93, 94) be a straight line, and  $ABCD$  any broken line *in space*, connecting the points  $A$  and  $D$ , and let  $A', B', C', D'$  be the projections of  $A, B, C, D$  upon  $OX$ , whose positive direction is  $OX$ . Denote the angle  $RAD$  by  $\phi$ , the length of the lines  $AB, BC, CD$  by  $l_1, l_2, l_3$ , and the angles which they make with the positive direction of  $OX$  by  $\alpha_1, \alpha_2, \alpha_3$ ; then in (Fig. 93), we have

$$A'D' = A'B' + B'C' + C'D';$$

Therefore  $AD \cos \phi = l_1 \cos \alpha_1 + l_2 \cos \alpha_2 + l_3 \cos \alpha_3$ . [56]

In Fig. 94,  $C'D'$  is negative; but  $l_3 \cos \alpha_3$  is also negative, since  $\alpha_3$ , or  $SCD$ , is obtuse; hence formula [56] holds true in all cases.

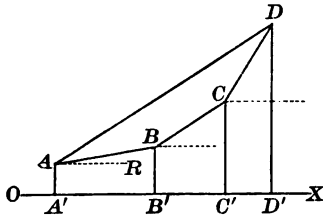


Fig. 93.

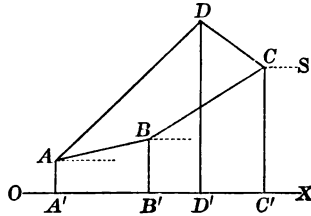


Fig. 94.

That is, the algebraic sum of the projections on a given line of the parts of any broken line connecting any two points is equal to the projection on the same line of the straight line joining the same two points.

217. To find the angle between two straight lines in terms of their direction cosines.

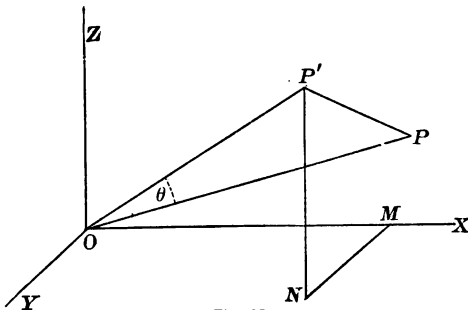


Fig. 95.

Let  $OP$  and  $OP'$  be parallel respectively to any two given lines in space. Let  $\theta$  denote their included angle, and  $\alpha, \beta, \gamma$ ,

and  $\alpha'$ ,  $\beta'$ ,  $\gamma'$  their direction angles, respectively. Let  $OM$ ,  $MN$ ,  $NP'$  be the co-ordinates of  $P'$ ; then the projection of  $OP'$  on  $OP$  equals the sum of the projections of  $OM$ ,  $MN$ ,  $NP'$  on  $OP$ ; that is,

$$OP' \cos \theta = OM \cos \alpha + MN \cos \beta + NP' \cos \gamma.$$

But  $OM = OP' \cos \alpha'$ ,  $MN = OP' \cos \beta'$ ,  $NP' = OP' \cos \gamma'$ .

$$\begin{aligned} \text{Hence } OP' \cos \theta &= OP' \cos \alpha' \cos \alpha + OP' \cos \beta' \cos \beta \\ &\quad + OP' \cos \gamma' \cos \gamma, \end{aligned}$$

$$\text{or } \cos \theta = \cos \alpha \cos \alpha' + \cos \beta \cos \beta' + \cos \gamma \cos \gamma', \quad [57]$$

which is the required formula.

**218.** To find the distance between two points in terms of their co-ordinates.

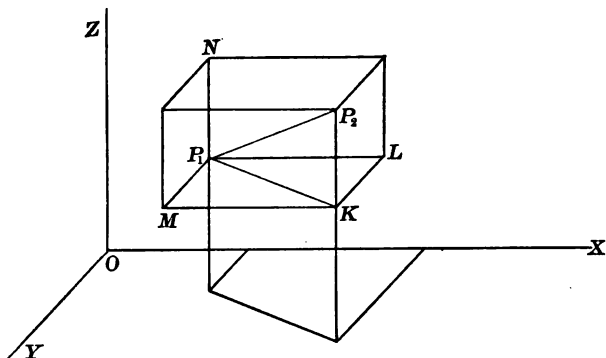


Fig. 96.

Let  $P_1$  be the point  $(x_1, y_1, z_1)$ , and  $P_2$  the point  $(x_2, y_2, z_2)$ . Through  $P_1$  and  $P_2$  pass planes parallel to the co-ordinate planes, thus forming the rectangular parallelepiped whose diagonal is  $P_1P_2$ , and whose edges  $P_1L$ ,  $LK$ ,  $KP_2$  are parallel to the axes of  $x$ ,  $y$ ,  $z$ , respectively.

$$\text{Then } P_1P_2^2 = P_1L^2 + LK^2 + KP_2^2. \quad (1)$$

Now  $P_1L$  is the difference of the distances of  $P_1$  and  $P_2$  from the plane  $yz$ , so that  $P_1L = x_2 - x_1$ . For like reason,  $LK = y_2 - y_1$ , and  $KP_2 = z_2 - z_1$ . Hence, denoting the distance  $P_1P_2$  by  $D$ , we have, by substituting in (1),

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}, \quad [58]$$

which is the required formula.

COR. 1. Since  $P_1L$ ,  $LK$ ,  $KP_2$  are equal to the projections of the line  $P_1P_2$  on the co-ordinate axes, it follows, from [58], that

*The square of any line is equal to the sum of the squares of its projections on the axes.*

COR. 2. If  $\alpha$ ,  $\beta$ ,  $\gamma$  be the direction angles of the line  $P_1P_2$ , we have

$$x_2 - x_1 = D \cos \alpha, \quad y_2 - y_1 = D \cos \beta, \quad z_2 - z_1 = D \cos \gamma.$$

**219. Polar Co-ordinates.** Let  $XOY$  be a fixed plane,  $OX$  a fixed line in it, and  $OZ$  a perpendicular to it at the fixed point  $O$ . To  $P$ , any point in space, draw  $OP$ , and through  $OP$

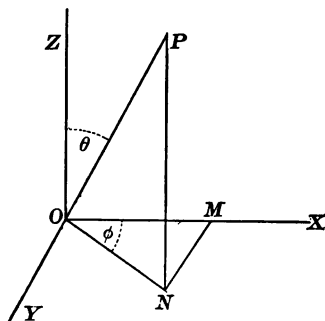


Fig. 97.

pass a plane perpendicular to  $XOY$ , intersecting the latter in  $ON$ ; then the distance  $OP$  and the angles  $ZOP$  and  $MON$  determine the point  $P$ , and are called its **Polar Co-ordinates**;

$OP$ , denoted by  $\rho$ , is the **Radius Vector**; and the angles  $ZOP$  and  $MON$ , denoted by  $\theta$  and  $\phi$  respectively, are the **Vectorial Angles**. The point  $P$  is written  $(\rho, \theta, \phi)$ .  $\phi$  determines the plane  $ZON$ ,  $\theta$  determines the line  $OP$  in that plane, and  $\rho$  locates  $P$  in  $OP$ .

**COR.** If  $XOY$  be a right angle, the rectangular co-ordinates of  $P$  are  $OM$ ,  $MN$ ,  $NP$ . To express these in terms of the polar co-ordinates of  $P$ , we have

$$x = OM = ON \cos \phi = OP \sin \theta \cos \phi = \rho \sin \theta \cos \phi.$$

$$y = MN = ON \sin \phi = OP \sin \theta \sin \phi = \rho \sin \theta \sin \phi.$$

$$z = NP = OP \cos \theta = \rho \cos \theta.$$

We readily obtain also

$$\rho = \sqrt{(x^2 + y^2 + z^2)},$$

$$\tan \theta = \frac{\sqrt{(x^2 + y^2)}}{z}, \quad \tan \phi = \frac{y}{x}.$$

**220.** The **Projection** of a point on a plane is the foot of the perpendicular from the point to the plane. The perpendicular itself is the **Projector** of the point. Thus, the point  $N$  (Fig. 97) is the projection of  $P$  on the plane  $xy$ , and  $PN$  is its projector.

The *projection* of a limited straight line on a plane is the straight line joining the projections of its extremities. The **Inclination** of a line to a plane is the angle which it makes with its projection on that plane. The projection of a limited line is evidently equal to its length multiplied by the cosine of its inclination. Thus,  $ON = OP \cos NOP$ .

The *projection* of any curve upon a plane is the locus of the projections of all its points. The **Projecting Cylinder** of a curve is the locus of the projectors of all its points. In the case of a right line this locus is the **Projecting Plane**.

## Ex. 43.

1. Find the distance between the points  $(1, 2, 3)$ ,  $(2, 3, 4)$ ;  $(2, 3, 4)$ ,  $(3, 4, 5)$ ;  $(1, 2, 3)$ ,  $(3, 4, 5)$ .

2. Prove that the triangle formed by joining the three points  $(1, 2, 3)$ ,  $(2, 3, 1)$ ,  $(3, 1, 2)$  is equilateral.

3. The lengths of the projections of a line on the three co-ordinate axes are 3, 4, 5, respectively; find the length of the line.

4. Find the direction cosines of the radius vector of the point  $(-3, -4, 5)$ .

5. What lines have direction cosines proportional to 3, -2, -5? Find the values of these direction cosines.

6. Find the angle between two straight lines whose direction cosines are proportional to 1, 2, 3, and 2, 3, 6, respectively.

7. Find the angle between two straight lines whose direction cosines are proportional to 1, 2, 3, and 5, -4, 1, respectively.

8. Find the polar co-ordinates of the point  $(\sqrt{3}, 1, 2\sqrt{3})$ .

9. Find the rectangular co-ordinates of the point  $(4, \frac{1}{3}\pi, \frac{1}{3}\pi)$ .

10. If  $(x, y, z)$  bisect the line joining  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$ , prove that  $x = \frac{1}{2}(x_1 + x_2)$ ,  $y = \frac{1}{2}(y_1 + y_2)$ ,  $z = \frac{1}{2}(z_1 + z_2)$ .

11. If  $(x, y, z)$  divide the line joining  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$ , in the ratio,  $m_1 : m_2$ ; prove that

$$x = \frac{m_2 x_1 + m_1 x_2}{m_2 + m_1}, \quad y = \frac{m_2 y_1 + m_1 y_2}{m_2 + m_1}, \quad z = \frac{m_2 z_1 + m_1 z_2}{m_2 + m_1}.$$

12. Find the co-ordinates of the point that divides the line joining  $(3, -2, 4)$  and  $(1, 3, -2)$  in the ratio 1 : 3.

13. Find the point that divides the line joining  $(-2, -3, -1)$  and  $(-5, -2, 4)$  in the ratio 5 : 2.



## CHAPTER II.

### THE PLANE.

**221.** *To find the equation of a plane in terms of the length of the perpendicular from the origin, and its direction cosines.*

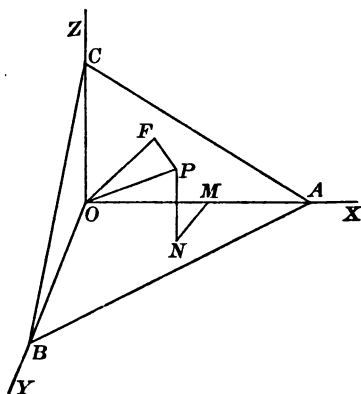


Fig. 98.

Let  $OF$  be the perpendicular to the plane  $ABC$  from the origin  $O$ ; denote its length by  $p$ , and its direction angles by  $\alpha, \beta, \gamma$ . Let  $P$  be any point in the plane,  $OP$  its radius vector, and  $OM, MN, NP$  its co-ordinates,  $x, y, z$ . Then the projection of  $OP$  on  $OF$  is equal to the sum of the projections of  $OM, MN, NP$  on  $OF$ . But as the plane is perpendicular to  $OF$ ,  $p$  is the projection of  $OP$  on  $OF$ ; and the projections of  $OM, MN, NP$  on  $OF$  are respectively  $x \cos \alpha, y \cos \beta, z \cos \gamma$ ; hence

$$x \cos \alpha + y \cos \beta + z \cos \gamma = p, \quad [59]$$

which is the equation required. Equation [59] is called the *normal* equation of a plane.

COR. 1. When the plane is perpendicular to one of the co-ordinate planes, the plane  $xy$  for example,  $OF$  lies in the plane  $xy$ ; hence  $\gamma = \frac{1}{2}\pi$ ,  $\cos \gamma = 0$ , and equation [59] becomes

$$x \cos \alpha + y \cos \beta = p. \quad (2)$$

COR. 2. When the plane is parallel to one of the co-ordinate planes, as the plane  $yz$ ,  $OF$  lies in the axis of  $x$ ; hence  $\cos \alpha = 1$ ,  $\cos \beta = 0$ ,  $\cos \gamma = 0$ , and [59] becomes

$$x = p. \quad (3)$$

COR. 3. Since  $OF$  is perpendicular to the plane  $ABC$ , and  $OX$  to  $YOZ$ , the dihedral angle  $A-BC-O = \text{angle } FOX$ . For like reason  $B-CA-O = FOY$ , and  $C-BA-O = FOZ$ .

**222.** *The locus of every equation of the first degree between three variables is a plane.*

A general form embracing every equation of the first degree between  $x, y, z$ , is

$$Ax + By + Cz = D, \quad (1)$$

in which  $D$  is positive. Dividing both members by

$$\sqrt{A^2 + B^2 + C^2},$$

we obtain

$$\begin{aligned} \frac{A}{\sqrt{A^2 + B^2 + C^2}}x + \frac{B}{\sqrt{A^2 + B^2 + C^2}}y + \frac{C}{\sqrt{A^2 + B^2 + C^2}}z \\ = \frac{D}{\sqrt{A^2 + B^2 + C^2}}, \end{aligned} \quad (2)$$

in which the coefficients of  $x, y, z$ , are the direction cosines of some line (§ 213, Cor.). Thus (2) is in the form of [59] § 221; hence the locus of (2), or (1), is a plane.

COR. 1. The length of the perpendicular from the origin upon plane (1) equals the second member of equation (2), and

the direction cosines of this perpendicular are the coefficients in (2) of  $x, y, z$ , respectively. These direction cosines are evidently proportional to  $A, B, C$ .

Hence, to construct equation (1), draw the radius vector of the point  $(A, B, C)$ ; the plane perpendicular to this line at the distance  $\frac{D}{\sqrt{A^2 + B^2 + C^2}}$  from the origin is the locus of (1).

COR. 2. To reduce any simple equation to the *normal form*, put it in the form of  $Ax + By + Cz = D$ , in which  $D$  is positive, and divide both members by  $\sqrt{A^2 + B^2 + C^2}$ .

COR. 3. If a simple equation contain only two variables, its locus is perpendicular to the corresponding co-ordinate plane; if only one variable, its locus is perpendicular to the corresponding co-ordinate axis.

**223.** *To find the equation of a plane in terms of its intercepts on the axes.*

Let  $a, b, c$  denote respectively the intercepts on the axes of the plane whose equation is

$$Ax + By + Cz = D. \quad (1)$$

Making  $y = z = 0$ , and therefore  $x = a$ , (1) becomes

$$Aa = D, \text{ or } A = D \div a.$$

Making  $x = z = 0$ , and therefore  $y = b$ , (1) becomes

$$Bb = D, \text{ or } B = D \div b.$$

Making  $x = y = 0$ , and therefore  $z = c$ , (1) becomes

$$Cc = D, \text{ or } C = D \div c.$$

Substituting these values in (1), and dividing by  $D$ , we have

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1, \quad [60]$$

which is the required equation. Equation [60] is called the *symmetrical equation of a plane*.

**224.** *To find the angle between any two planes.*

The angle included between the two planes

$$Ax + By + Cz = D,$$

$$A'x + B'y + C'z = D',$$

is evidently equal to the angle included between the perpendiculars to them from the origin. But the direction cosines of these perpendiculars are respectively (§ 222, Cor. 1).

$$\frac{A}{\sqrt{A^2 + B^2 + C^2}}, \quad \frac{B}{\sqrt{A^2 + B^2 + C^2}}, \quad \frac{C}{\sqrt{A^2 + B^2 + C^2}},$$

$$\frac{A'}{\sqrt{A'^2 + B'^2 + C'^2}}, \quad \frac{B'}{\sqrt{A'^2 + B'^2 + C'^2}}, \quad \frac{C'}{\sqrt{A'^2 + B'^2 + C'^2}}.$$

Substituting these values in [57], we have

$$\cos \theta = \frac{AA' + BB' + CC'}{\sqrt{A^2 + B^2 + C^2} \sqrt{A'^2 + B'^2 + C'^2}}, \quad [61]$$

in which  $\theta$  equals the angle included between the planes.

COR. 1. If the planes are parallel to each other,  $\theta = 0$  and  $\cos \theta = 1$ . Putting  $\cos \theta = 1$  in [61], clearing of fractions, squaring, transposing, and uniting, we obtain

$$(AB' - BA')^2 + (AC' - CA')^2 + (BC' - CB')^2 = 0.$$

Each term being a square, and therefore positive, this equation can be satisfied only when each term equals zero, giving us

$$AB' = BA', \quad AC' = CA', \quad BC' = CB',$$

or 
$$\frac{A}{A'} = \frac{B}{B'} = \frac{C}{C'}.$$

Hence, if two planes are parallel, the coefficients of  $x, y, z$ , in their equations, are proportional, and conversely.

COR. 2. If the planes are perpendicular to each other,  $\cos \theta = 0$ ; and hence

$$AA' + BB' + CC' = 0,$$

and conversely.

225. To find the perpendicular distance of a given point from a given plane.

Let the equation of the given plane be

$$x \cos \alpha + y \cos \beta + z \cos \gamma = p, \quad (1)$$

and let  $(x_1, y_1, z_1)$  be the given point. Let the plane

$$x \cos \alpha + y \cos \beta + z \cos \gamma = p', \quad (2)$$

which is evidently parallel to the given plane, pass through the given point  $(x_1, y_1, z_1)$ ; then we have

$$x_1 \cos \alpha + y_1 \cos \beta + z_1 \cos \gamma = p'. \quad (3)$$

Hence  $x_1 \cos \alpha + y_1 \cos \beta + z_1 \cos \gamma - p = p' - p$ .

But  $p' - p$  equals numerically the distance between the planes (1) and (2), and is therefore the required distance.

Hence, to find the distance of any point from the plane

$$x \cos \alpha + y \cos \beta + z \cos \gamma = p,$$

substitute the co-ordinates of the point for  $x, y, z$  in the expression

$$x \cos \alpha + y \cos \beta + z \cos \gamma - p.$$

COR. If the equation of the plane be  $Ax + By + Cz = D$ , and  $d$  denote the distance of  $(x_1, y_1, z_1)$  from this plane, we have

$$d = \frac{Ax_1 + By_1 + Cz_1 - D}{\sqrt{A^2 + B^2 + C^2}}.$$

The distance as given by the formulas will evidently be *positive* or *negative*, according as the point and origin are on opposite sides of the plane, or on the same side. The sign may be neglected if simply the numerical distance is required.

#### EX. 44.

1. To which co-ordinate plane is  $3y - 4z = 2$  perpendicular?  
 $x - 8z - 7 = 0$ ?  $x - 2y = 2$ ?  $x = mz + p$ ?  $y = nx + q$ ?  
 What is the locus of  $z = 5$ ?  $y = -7$ ?  $y = 4$ ?  $z = -2$ ?  
 $x = 0$ ?  $y = 0$ ?  $z = 0$ ?

2. Reduce to the normal form

$$3x - 2y + z = 2; \quad 5x - 4y + z = 4.$$

What is the distance of each of these planes from the origin? What are the direction cosines of the perpendiculars to each? Which of the eight octants does each truncate?

3. Find the intercepts on the axes of  $3x - 2y + 4z - 12 = 0$ ; of  $6x - 4y - 3z + 24 = 0$ ;  $5x + 7y + 5z + 35 = 0$ . Which of the eight octants does each truncate?

Reduce each equation to the symmetrical form.

4. What is the equation of the plane at the distance 7 from the origin, and perpendicular to a line whose direction cosines are proportional to 2, -3, and  $\sqrt{3}$ ?

5. What is the equation of the plane whose intercepts on the axes are respectively 4, -3, -7? -1, -2, -5?  $\frac{1}{2}$ ,  $-\frac{2}{3}$ ,  $\frac{1}{4}$ ?

6. Find the equation of the plane passing through the points (1, 2, 3), (0, 4, -1), and (1, -1, 0).

7. Find the angle between the planes

$$2x + z - y = 3,$$

$$z + x + 2y = 5.$$

8. Find the angle between the planes

$$3z + 5x - 7y = -1,$$

$$3z - 2x - y = 0.$$

9. Find the angle which the plane  $Ax + By + Cz = D$  makes with each of the co-ordinate planes.

10. Find the distance from the point (2, -3, 0) to the plane

$$\sqrt{3}z + 2x - 3y = 4.$$

11. Show that the two points (1, -1, 3) and (3, 3, 3) are on opposite sides of, and equidistant from, the plane

$$5x + 2y - 7z + 9 = 0.$$

12. If, in Fig. 92,  $OM = a$ ,  $OR = b$ ,  $OS = c$ , find the equation of the plane through the points  $M$ ,  $P$ ,  $R$ . Find the length of the perpendicular from  $S$  upon this plane.

13. Prove that the plane

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$$

passes through the point  $(x_1, y_1, z_1)$ , and is parallel to the plane  $Ax + By + Cz = D$ .

14. Find the equation of a plane passing through the point  $(3, 4, -1)$ , and parallel to the plane  $2x + 4y - z = 2$ .

15. What three equations must be satisfied in order that the plane  $Ax + By + Cz = D$  may pass through the two points  $(x_1, y_1, z_1)$ ,  $(x_2, y_2, z_2)$ , and be perpendicular to the plane

$$A'x + B'y + C'z = D'?$$

16. Find the equation of the plane passing through the points  $(1, 1, 1)$ ,  $(2, 0, -1)$ , and perpendicular to the plane  $x + y - z = 3$ .

17. What three equations must be satisfied, in order that the plane  $Ax + By + Cz = D$  may pass through the three points  $(x_1, y_1, z_1)$ ,  $(x_2, y_2, z_2)$ ,  $(x_3, y_3, z_3)$ ?

18. Find the equation of the plane which passes through the points  $(1, 2, 3)$ ,  $(3, 2, 1)$ ,  $(2, 3, 1)$ , and find the distance of this plane from the origin.

19. Find the equation of the plane through  $(2, 3, -1)$  parallel to the plane  $3x - 4y + 7z = 0$ .

20. Find the equation of a plane which passes through the point  $(1, 2, 3)$ , and is perpendicular to each of the planes  $x + 2z = 1$ ,  $y + 5z = 1$ .

## CHAPTER III.

### THE STRAIGHT LINE.

**226.** *To find the equations of a straight line.*

The co-ordinates of any point on the line of intersection of two planes will satisfy the equation of each of these planes. Hence, any two simultaneous equations of the first degree in  $x$ ,  $y$ , and  $z$  represent some straight line. Of the indefinite

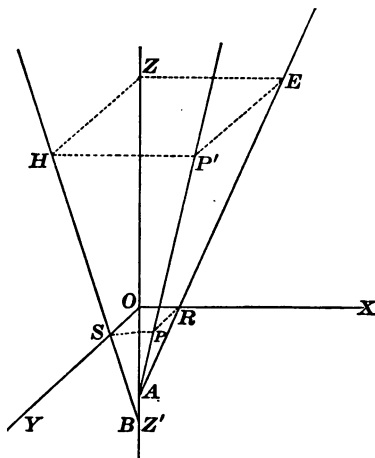


Fig. 99.

number of pairs of planes that intersect in, and therefore determine, a straight line, the equations of its projecting planes on the co-ordinate planes are the simplest, and two of them are taken as the equations of the line. Thus let  $PP'ER$  and  $PP'HS$  be the projecting planes of any straight line  $PP'$



on the co-ordinate planes  $xz$  and  $yz$ , respectively; and let the equations of these projecting planes be

$$\begin{cases} x = mz + p, & (1) \\ y = nz + q; & (2) \end{cases}$$

then are (1) and (2) the equations of the line  $PP'$ .

COR. Let  $RE$  and  $SH$  be the *projections* of the line  $PP'$  on the planes  $xz$  and  $yz$ , respectively. Since the line  $RE$  lies in the plane  $PP'RE$ , equation (1) expresses the relation between the co-ordinates  $x, z$ , of every point in  $RE$ ; hence (1) is the equation of  $RE$  referred to the axes  $ZZ'$  and  $OX$ . For like reason (2) is the equation of the projection  $SH$  referred to the axes  $ZZ'$  and  $OY$ .

$$\begin{aligned} \text{Hence} \quad m &= \tan ZAE = \text{slope } RE; \\ p &= OR = \text{intercept of } RE \text{ on } OX; \\ n &= \tan ZBH = \text{slope } SH; \\ q &= OS = \text{intercept of } SH \text{ on } OY. \end{aligned}$$

REM. The locus of (1) in space is the plane  $PP'ER$ , while its locus in the plane  $xz$  is the line  $RE$ . Similarly, the locus of (2) in space is the plane  $PP'HS$ , while its plane locus is  $SH$ . The locus in space of (1) and (2), considered as simultaneous, is the line  $PP'$ .

COR. 1. Eliminating  $z$  between (1) and (2), we obtain

$$y = \frac{n}{m}x + \left(q - \frac{np}{m}\right)$$

whose locus in space is the projecting plane of  $PP'$  on the plane  $xy$ , and whose locus in the plane  $xy$  is the projection of  $PP'$  on that plane.

COR. 2. Making  $z = 0$  in equations (1) and (2), we obtain

$$x = p, \quad y = q;$$

hence the line  $PP'$  pierces the plane  $xy$  in the point  $(p, q, 0)$ . This is evident also from the figure.

In like manner, we find that the line pierces the planes  $xz$  and  $yz$  respectively in the points

$$\left(\frac{np - mq}{n}, 0, -\frac{q}{n}\right), \left(0, \frac{mq - np}{m}, -\frac{p}{m}\right).$$

**227.** To find the symmetrical equations of a right line.

Let  $\alpha, \beta, \gamma$  be the direction angles of any right line,  $(x_1, y_1, z_1)$  some fixed point in it, and  $(x, y, z)$  any other point of the line. Let  $r$  denote the distance between these two points. Then by § 218, Cor. 2, we have

$$x - x_1 = r \cos \alpha, \quad y - y_1 = r \cos \beta, \quad z - z_1 = r \cos \gamma. \quad (1)$$

$$\text{Whence} \quad \frac{x - x_1}{\cos \alpha} = \frac{y - y_1}{\cos \beta} = \frac{z - z_1}{\cos \gamma}, \quad [63]$$

which are the symmetrical equations of a right line passing through the point  $(x_1, y_1, z_1)$ .

COR. If [63] passes through a second point  $(x_2, y_2, z_2)$ , its co-ordinates must satisfy [63]; hence, we have

$$\frac{x_2 - x_1}{\cos \alpha} = \frac{y_2 - y_1}{\cos \beta} = \frac{z_2 - z_1}{\cos \gamma}. \quad (2)$$

Dividing each member of [63] by the corresponding member of (2), we obtain

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}, \quad [64]$$

which are the equations of a right line through the two points  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$ .

**228.** If we divide the denominators in any equations of the form

$$\frac{x - x_1}{L} = \frac{y - y_1}{M} = \frac{z - z_1}{N} \quad (1)$$

by  $\sqrt{L^2 + M^2 + N^2}$ , the denominators will then be the direction cosines of some line (§ 213, Cor.), and the equations will be in the form of [63].

Hence, to reduce equations in the form of (1) to the symmetrical form, *divide each denominator by the square root of the sum of their squares.*

COR. The locus of equations (1) is the line through  $(x_1, y_1, z_1)$  parallel to the radius vector of the point  $(L, M, N)$ .

229. To find the angle between the lines

$$\frac{x - x_1}{L} = \frac{y - y_1}{M} = \frac{z - z_1}{N},$$

and

$$\frac{x - x_2}{L'} = \frac{y - y_2}{M'} = \frac{z - z_2}{N'}.$$

By § 228 the direction cosines of these lines are respectively

$$\frac{L}{\sqrt{L^2 + M^2 + N^2}}, \quad \frac{M}{\sqrt{L^2 + M^2 + N^2}}, \quad \frac{N}{\sqrt{L^2 + M^2 + N^2}};$$

$$\frac{L'}{\sqrt{L'^2 + M'^2 + N'^2}}, \quad \frac{M'}{\sqrt{L'^2 + M'^2 + N'^2}}, \quad \frac{N'}{\sqrt{L'^2 + M'^2 + N'^2}}.$$

Substituting these values in [57], we obtain

$$\cos \theta = \frac{LL' + MM' + NN'}{\sqrt{L^2 + M^2 + N^2} \sqrt{L'^2 + M'^2 + N'^2}}. \quad [65]$$

COR. 1. If the lines are parallel,  $\frac{L}{L'} = \frac{M}{M'} = \frac{N}{N'}$ , and conversely.

COR. 2. If the lines are perpendicular,

$$LL' + MM' + NN' = 0,$$

and conversely.

230. To find the inclination of the line

$$\frac{x - x_1}{L} = \frac{y - y_1}{M} = \frac{z - z_1}{N} \quad (1)$$

$$\text{to the plane } Ax + By + Cz = D. \quad (2)$$

The equation of the perpendicular from  $(x_1, y_1, z_1)$  to the plane is

$$\frac{x - x_1}{A} = \frac{y - y_1}{B} = \frac{z - z_1}{C}. \quad (3)$$

Now, the inclination of line (1) to plane (2) is evidently the complement of the angle between the lines (1) and (3). Denote this inclination by  $v$ ; then  $\sin v = \cos \theta$ ,  $\theta$  being the angle between the lines (1) and (3). Hence

$$\sin v = \frac{AL + BM + CN}{\sqrt{A^2 + B^2 + C^2} \sqrt{L^2 + M^2 + N^2}}. \quad [66]$$

COR. 1. If the line is parallel to the plane,  $\sin v = 0$ , and therefore  $AL + BM + CN = 0$ , and conversely.

COR. 2. If the line is perpendicular to the plane,  $\sin v = 1$ , and therefore  $\frac{L}{A} = \frac{M}{B} = \frac{N}{C}$ , and conversely.

COR. 3. If line (1) lies in plane (2); then

$$AL + BM + CN = 0,$$

$$\text{and} \quad Ax_1 + By_1 + Cz_1 = D,$$

and conversely.

#### Ex. 45.

1. Determine the position, direction cosines, and direction angles of the intersection of the planes  $x + y - z + 1 = 0$ , and  $4x + y - 2z + 2 = 0$ .

Eliminating successively  $y$  and  $z$  between the equations, we obtain  $3x - z + 1 = 0$  and  $2x - y = 0$ ; or  $\frac{x}{1} = \frac{y}{2} = \frac{z - 1}{3}$ .

From the last form we know that the line passes through the point  $(0, 0, 1)$ , and is parallel to the radius vector of the point  $(1, 2, 3)$ . The direction cosines are found by dividing the denominators 1, 2, 3, by  $\sqrt{14}$ ; and the direction angles are found from their cosines.

2. Determine the position and direction cosines of the intersection of  $x - 2y = 5$ , and  $3x + y - 7z = 0$ .

Here  $\frac{x-5}{2} = \frac{y}{1} = \frac{z-\frac{15}{7}}{1}$ , whence the line passes through the point  $(5, 0, \frac{15}{7})$ , and is parallel to the radius vector of  $(2, 1, 1)$ .

3. Determine the position of the line

$$5x - 4y = 1, \quad 3y - 5z = 2.$$

4. What is the position of the line  $x = 3, y = 4$ ? Of the line  $y = 4, z = -5$ ? Of the line  $x = -2, z = 3$ ?

5. Find the equations of the right line passing through the points  $(1, 2, 3), (3, 4, 1)$ .

6. Find the points in which the line of Example 5 pierces the co-ordinate planes.

7. Two of the projecting planes of a line are  $x + y = 4$ , and  $2x - 5z = -2$ ; find the third.

8. A line passes through  $(2, 1, -1)$  and  $(-3, -1, 1)$ ; find the equations of its projections on the co-ordinate planes.

9. Show that the lines  $\frac{x}{1} = \frac{y}{2} = \frac{z}{1}$ , and  $\frac{x}{1} = \frac{y}{-1} = \frac{z}{1}$  are at right angles.

10. Show that the line  $4x = 3y = -z$  is perpendicular to the line  $3x = -y = -4z$ .

11. Find the angle between the lines

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{0}, \text{ and } \frac{x}{3} = \frac{y}{-4} = \frac{z}{5}.$$

12. Find the angle between the right lines

$$y = 5x + 3, \quad z = 3x + 5,$$

and  $y = 2x, \quad z = x + 1.$

13. Find the angle between the lines

$$y = 2x + 2, \quad z = 2x + 1,$$

and  $y = 4x + 1, \quad z = x + 5.$

14. Show that the lines

$$3x + 2y + z - 5 = 0, \quad x + y - 2z - 3 = 0,$$

and  $8x - 4y - 4z = 0, \quad 7x + 10y - 8z = 0,$

are at right angles.

15. Find the equations of the line through  $(-2, 3, -1)$  parallel to the line  $y = -2x + 1, z = 3x - 4$ .

16. Find the equations of the right line through  $(3, -7, -5)$ , its direction cosines being proportional to  $-3, 5, -6$ .

17. Find the equations of the line through  $(2, -4, -6)$  perpendicular to the plane  $3x - 6y + 2z = 4$ .

18. Find the inclination of the line

$$\frac{x-4}{3} = \frac{y+2}{-2} = \frac{z-5}{-4}$$

to the plane  $2x - 4y + 3z = 1$ .

19. Reduce the equations  $x = mz + p, y = nz + q$ , to the symmetrical form, and thus find the direction cosines in terms of  $m$  and  $n$ .

20. Show that the formula for the angle included between the lines

$$x = mz + p, \quad y = nz + q,$$

and  $x = m'z + p', \quad y = n'z + q',$

is 
$$\cos \theta = \frac{mm' + nn' + 1}{\sqrt{m^2 + n^2 + 1} \sqrt{m'^2 + n'^2 + 1}}.$$

21. Prove that the lines in Example 20 are perpendicular if  $mm' + nn' + 1 = 0$ , and conversely. Prove that they are parallel if  $m = m'$  and  $n = n'$ , and conversely.

22. Prove that two lines are parallel if their projections are parallel, and conversely.

## SUPPLEMENTARY PROPOSITIONS.

**231.** The **Traces** of a plane are its lines of intersection with the co-ordinate planes. Thus  $AB$ ,  $BC$ ,  $CA$  (Fig. 98), are the traces of the plane  $ABC$ .

**232.** *To find the equations of the traces of the plane*

$$Ax + By + Cz = D. \quad (1)$$

For every point in the plane  $xy$ ,  $z = 0$ ; hence, putting  $z = 0$  in (1), we obtain

$$Ax + By = D, \quad (2)$$

which is the equation of the trace  $AB$  (Fig. 98) on the plane  $xy$ . For like reason

$$By + Cz = D, \quad (3)$$

$$\text{and} \quad Ax + Cz = D, \quad (4)$$

are the equations of the traces  $BC$  and  $CA$  on the planes  $yz$  and  $xz$  respectively.

**COR.** The perpendicular from the origin to (1) is

$$\frac{x}{A} = \frac{y}{B} = \frac{z}{C},$$

and its projections on the co-ordinate planes are

$$Bx = Ay, \quad Cy = Bz, \quad Cx = Az. \quad (5)$$

By comparing coefficients, we see that lines (5) are perpendicular to (2), (3), and (4), respectively. Hence,

*If a line in space is perpendicular to a plane, its projections are perpendicular to the traces of the plane.*

**233.** *To find the condition that the right lines*

$$x = mz + p, \quad y = nz + q,$$

$$\text{and} \quad x = m'z + p', \quad y = n'z + q',$$

*may intersect, and to find their points of intersection.*

Equating the two values of  $x$  and of  $y$ , we have

$$z = \frac{p' - p}{m - m'}, \quad z = \frac{q' - q}{n - n'}.$$

If the two lines intersect, these two values of  $z$  must be equal ;

hence 
$$\frac{p' - p}{m - m'} = \frac{q' - q}{n - n'}$$

is the equation of condition that the two given lines in space intersect.

When this condition is fulfilled, the values of  $x$  and  $y$  may be found by substituting either value of  $z$  in the equations of either line.

**234.** To pass a plane through the point  $(x_1, y_1, z_1)$  and the right line

$$\frac{x - x_1}{L} = \frac{y - y_1}{M} = \frac{z - z_1}{N}. \quad (a)$$

If the plane  $Ax + By + Cz = D$  (1)

pass through the point  $(x_1, y_1, z_1)$ , we have

$$Ax_1 + By_1 + Cz_1 = D; \quad (2)$$

and if line (a) lies in plane (1), we have

$$Ax_1 + By_1 + Cz_1 = D, \quad (3)$$

and  $AL + BM + CN = 0. \quad (4)$

The equation of the required plane is found by eliminating  $A, B, C, D$  between (1), (2), (3), and (4).

To simplify the process of elimination, (1) might be written in the form  $A'x + B'y + C'z = 1$ , but the solution would be less general, as it would not embrace the case when  $D = 0$ .

**235.** From the forms  $x = mz + p, y = nz + q$ , show

That the equations of a line passing through  $(x_1, y_1, z_1)$  are

$$\begin{cases} x - x_1 = m(z - z_1). \\ y - y_1 = n(z - z_1). \end{cases}$$



## CHAPTER IV.

### SURFACES OF REVOLUTION.

**236.** It has been shown that a single equation of the first degree between three variables represents a plane surface, and that two such equations in general represent a right line. It is evident, moreover, that in general three such equations determine a point common to their loci. Thus, if in Fig. 92,  $OM = a$ ,  $OR = b$ , and  $OS = c$ ; then the equations  $x = a$ ,  $y = b$ ,  $z = c$  determine the point  $P$ , and are called its equations. We proceed to show that

*In general, any single equation of the form  $f(x, y, z) = 0$ , represents a surface of some kind; two such equations represent a curve, and three determine one or more points.*

(i.) Let two of the variables be absent; for example, let the equation be  $f(x) = 0$ . Now  $f(x) = 0$  may be written in the form

$$(x - a_1)(x - a_2)(x - a_3) \dots (x - a_n) = 0 \quad (1)$$

in which  $a_1, a_2, a_3, \dots, a_n$  are the  $n$  roots of  $f(x) = 0$ . The locus of (1) is evidently the  $n$  parallel planes  $x = a_1, x = a_2, \dots, x = a_n$ . Similarly the equations  $f(y) = 0, f(z) = 0$  represent planes perpendicular to the axes of  $y$  and  $z$  respectively.

(ii.) Let one of the variables be absent; for example, let the equation be  $f(x, y) = 0$ . The locus of  $f(x, y) = 0$  in the plane  $xy$  is some plane curve. Through  $P$ , any point in this curve, conceive a line parallel to the axis of  $z$ ; then the coordinates  $x, y$  of all points in this line will equal those of  $P$ , and hence satisfy the equation  $f(x, y) = 0$ . Hence the locus

in space of  $f(x, y) = 0$  is the surface generated by a right line which is always parallel to the axis of  $z$ , and which moves along the plane locus of  $f(x, y) = 0$ .

That is, *the locus in space of  $f(x, y) = 0$  is a cylindrical surface whose elements are parallel to the axis of  $z$ , and whose directrix is the plane locus of  $f(x, y) = 0$ .*

Similarly the equations  $f(x, z) = 0$  and  $f(y, z) = 0$  represent cylindrical surfaces whose elements are parallel to the axes of  $y$  and  $x$  respectively.

(iii.) Let the equation be  $f(x, y, z) = 0$ . If in this equation we put  $x = a$  and  $y = b$ , the roots of the resulting equation in  $z$  will give the points in the locus that lie on the line through  $(a, b, 0)$  parallel to the axis of  $z$ . But as the number of these roots is finite, the number of points of the locus on this line is finite. Hence, the locus which embraces all such points for different values of  $a$  and  $b$  must be a surface and not a solid.

(iv.) Two equations considered as simultaneous are satisfied by the co-ordinates of all the points of intersection of their loci; that is, they represent the curve of intersection of two surfaces.

(v.) Three independent simultaneous equations are satisfied only by the co-ordinates of the points in which the curve represented by two of them cuts the surface represented by the third; hence they determine these points.

COR. 1. From (ii.) it follows that  $x^2 + y^2 = r^2$  is the equation of a cylinder whose axis is the axis of  $z$ , and whose radius is  $r$ . Also

$$y^2 = 4px, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \text{and} \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

are equations of cylindrical surfaces whose elements are parallel to the axis of  $z$ , and whose directrices are respectively the parabola, the ellipse, and the hyperbola.

**COR. 2.** If  $F(x, y) = 0$  is the equation obtained by eliminating  $z$  between the two equations  $f(x, y, z) = 0$ , and  $f_1(x, y, z) = 0$ ; then the locus in space of  $F(x, y) = 0$  is the projecting cylinder on the plane  $xy$  of the curve represented by the two equations. The *plane* locus of  $F(x, y) = 0$  is the projection of this curve on the plane  $xy$ . If the curve is parallel to the plane of projection, the curve and its projection are equal.

The equation obtained by eliminating  $x$  or  $y$  between the two equations has evidently a like interpretation.

**237.** The **Traces of a Surface** are its intersections with the co-ordinate planes.

If  $f(x, y, 0) = 0$  denote the equation obtained by making  $z = 0$  in  $f(x, y, z) = 0$ ; then the plane locus of  $f(x, y, 0) = 0$  is evidently the *trace* of the *surface*  $f(x, y, z) = 0$  on the plane  $xy$ .

### SURFACES OF REVOLUTION.

**238.** A **Surface of Revolution** is a surface that may be generated by a curve revolving about a fixed straight line as an axis. The revolving curve is called the **Generatrix**; and the fixed right line, the **Axis of Revolution**, or simply the **Axis**. A section of the surface made by a plane passing through the axis is called a **Meridian Section**. From these definitions, it follows that

(i.) Every section made by a plane perpendicular to the axis is a circle, whose centre is in the axis.

(ii.) Any meridian section is equal to the generatrix.

**239.** *To find the general equation of a surface of revolution.*

Let the axis of  $z$  be the axis of revolution, and let  $P$  be any point in the meridian section made by the plane  $xz$ . Let  $PHR$  be a section through  $P$  perpendicular to the axis of  $z$ , and denote the radius  $CH$ , or  $CP$ , of this circular section by  $r$ .

Now for all points in this circular section, we have  $x^2 + y^2 = r^2$ , and  $z = MP$ . The value of  $r^2$ , in terms of  $z$ , is obtained by substituting  $r$  for  $x$  in the equation of the meridian section made by the plane  $zx$ . Denoting this value of  $r^2$  by  $f(z)$ , and equating the two values of  $r^2$ , we have

$$x^2 + y^2 = f(z), \quad [67]$$

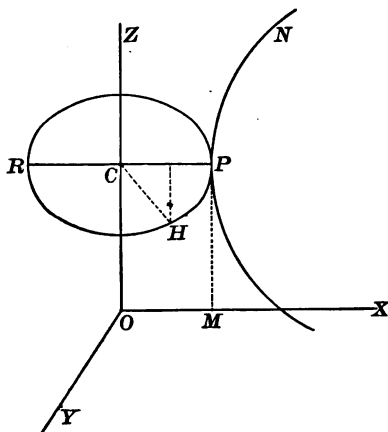


Fig. 100.

which expresses the relation between the co-ordinates  $x, y, z$  of all points in the section  $PHR$ . But as  $P$  is any point in the meridian section  $NP$ , [67] is the general equation of a surface of revolution whose axis is the axis of  $z$ .

**240. Paraboloid of Revolution.** A *Paraboloid of Revolution* is a surface that may be generated by a parabola revolving about its axis.

In this case the equation of the meridian section in the plane  $zx$  is

$$x^2 = 4pz;$$

hence

$$r^2 = 4pz = f(z).$$

Substituting in [67], we obtain

$$x^2 + y^2 = 4pz, \quad [68]$$

which is *the equation of the paraboloid of revolution*.

If in [68] we put  $x = m$ , we obtain

$$y^2 = 4pz - m^2, \quad (1)$$

which is the equation of the projection, on the plane  $yz$ , of the section of the paraboloid made by a plane parallel to the plane  $yz$ , and at a distance from it equal to  $m$ . Now the plane locus of (1), for all values of  $m$ , is a parabola; hence every plane section of the paraboloid parallel to the plane  $yz$  is a parabola. If in [68] we put  $y = n$ , we obtain

$$x^2 = 4pz - n^2. \quad (2)$$

From (2) we learn that all plane sections parallel to the plane  $xz$  are also parabolas. From definition, we know that all plane sections parallel to the plane  $xy$  are circles.

**241. Ellipsoid of Revolution.** An *Ellipsoid of Revolution*, or *Spheroid*, is a surface that may be generated by an ellipse revolving about one of its axes. It is called **Oblate** when the revolution is about the minor axis; and **Prolate** when about the major axis.

(i.) When the revolution is about the minor axis, the equation of the meridian section in the plane  $xz$  is

$$\frac{x^2}{a^2} + \frac{z^2}{b^2} = 1; \text{ hence } r^2 = a^2 \left( 1 - \frac{z^2}{b^2} \right) = f(z).$$

Substituting in [67], and reducing, we obtain

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1, \quad [69]$$

which is *the equation of the oblate spheroid*.

COR. 1. If  $a = b$ , [69] becomes

$$x^2 + y^2 + z^2 = a^2, \quad (1)$$

which is the equation of a sphere whose radius is  $a$ ,

Cor. 2. If in [69] we put  $x = m$ , we obtain

$$\frac{y^2}{a^2} + \frac{z^2}{b^2} = 1 - \frac{m^2}{a^2}. \quad (2)$$

Since (2) represents an ellipse, a point, or no locus in the plane  $yz$ , according as  $m^2 <, =, \text{ or } > a^2$ , we know that the surface lies between the two tangent planes  $x = a$  and  $x = -a$ , and that all plane sections parallel to the plane  $yz$  are ellipses.

If in [69] we put  $y = n$ , we obtain

$$\frac{x^2}{a^2} + \frac{z^2}{b^2} = 1 - \frac{n^2}{a^2}. \quad (3)$$

Since (3) represents an ellipse, a point, or no locus in the plane  $xz$ , according as  $n^2 <, =, \text{ or } > a^2$ , the surface lies between the two tangent planes  $y = a$  and  $y = -a$ , and all plane sections parallel to the plane  $xz$  are ellipses.

If in [69] we put  $z = q$ , we obtain

$$x^2 + y^2 = a^2 \left(1 - \frac{q^2}{b^2}\right). \quad (4)$$

Equation (4) represents a circle, a point, or no locus in the plane  $xy$ , according as  $q^2 <, =, > b^2$ . Hence the surface lies between the tangent planes  $z = b$  and  $z = -b$ , and all plane sections parallel to the plane  $xy$  are circles.

(ii.) When the revolution is about the major axis, the equation of the meridian section in the plane  $xz$  is

$$\frac{z^2}{a^2} + \frac{x^2}{b^2} = 1; \text{ hence } r^2 = b^2 \left(1 - \frac{z^2}{a^2}\right) = f(z).$$

Substituting in [67], we obtain

$$\frac{x^2}{b^2} + \frac{y^2}{b^2} + \frac{z^2}{a^2} = 1, \quad [70]$$

which is the equation of the prolate spheroid.

If in [69] we substitute  $a$  for  $b$  and  $b$  for  $a$ , we obtain [70]. Hence the discussion of [69] will apply to [70], if in that discussion  $b$  be substituted for  $a$  and  $a$  for  $b$ .

**242. Hyperboloid of Revolution.** A *Hyperboloid of Revolution* is a surface that may be generated by a hyperbola revolving about one of its axes. It consists of one or two nappes, or sheets, according as the hyperbola revolves about its conjugate or transverse axis.

(i.) If in [69] we substitute  $-b^2$  for  $b^2$ , we obtain

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} - \frac{z^2}{b^2} = 1, \quad [71]$$

which is the equation of the hyperboloid of one nappe.

If in [71] we put  $x = m$ , we have

$$\frac{y^2}{a^2} - \frac{z^2}{b^2} = 1 - \frac{m^2}{a^2}, \quad (1)$$

whose plane locus is a hyperbola for all values of  $m$ . Hence all plane sections parallel to the plane  $yz$  are hyperbolas. The transverse axis of any one of these hyperbolas is evidently parallel to the axis of  $y$  or  $z$ , according as  $m^2 < \text{or} > a^2$ . For  $m^2 = a^2$  (1) becomes

$$y = \pm \frac{a}{b} z.$$

Hence the sections of [71] made by the planes  $x = \pm a$  are each two intersecting right lines.

If in [71] we put  $y = n$ , we have

$$\frac{x^2}{a^2} - \frac{z^2}{b^2} = 1 - \frac{n^2}{a^2}. \quad (2)$$

Hence all plane sections of [71] parallel to the plane  $xz$  are hyperbolas, whose transverse axes are parallel to the axis of  $x$  or  $z$ , according as  $n^2 < \text{or} > a^2$ ; and the sections made by the planes  $y = \pm a$  are each two intersecting right lines.

If in [71] we put  $z = q$ , we obtain

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1 + \frac{q^2}{b^2},$$

whose plane locus is a circle for all values of  $q$ . This circle is smallest when  $q = 0$ . This smallest circle, which is the trace of the hyperboloid on the plane  $xy$ , is called the **Circle of the Gorge**.

(ii.) If in [70] we substitute  $-b^2$  for  $b^2$ , we obtain

$$\frac{x^2}{b^2} + \frac{y^2}{b^2} - \frac{z^2}{a^2} = -1, \quad [72]$$

which is *the equation of the hyperboloid of two nappes*.

The discussion of [72] for parallel plane sections is left as an exercise for the student.

**243.** The **Centre** of a surface is a point that bisects all chords passing through it.

**Central Surfaces** are such as have a centre.

The ellipsoids and hyperboloids of revolution are central surfaces. For, from their equations, it is evident that, if  $(x', y', z')$  be a point in any one of these surfaces,  $(-x', -y', -z')$  is also a point in the same surface. But the chord joining these two points is bisected by the origin, which is, therefore, the centre of the surface.

**244. Cone of Revolution.** A *Cone of Revolution* is a surface that may be generated by a right line revolving about an axis which it intersects.

Here the equation of the meridian section in plane  $xz$  is

$$z = mx + c;$$

$$\text{therefore} \quad r^2 = \left( \frac{z - c}{m} \right)^2 = f(z).$$

$$\text{Whence} \quad m^2(x^2 + y^2) = (z - c)^2 \quad [73]$$

is *the equation of the cone of revolution*.



In this equation  $c$  is the distance of the vertex from the origin and  $m = \tan XDB$ .

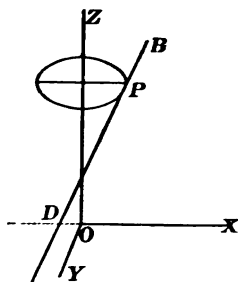


Fig. 101.

If  $c = 0$ , [73] becomes

$$m^2(x^2 + y^2) = z^2. \quad (1)$$

From (1) it is evident that the cone is a central surface.

If in (1) we put  $y = n$ , we obtain

$$\frac{z^2}{n^2 m^2} - \frac{x^2}{n^2} = 1,$$

whose plane locus is a hyperbola for all values of  $n$ . Hence all plane sections of the cone parallel to the plane  $zx$  are hyperbolas whose transverse axes are parallel to the axis of the cone. In like manner, we find that all plane sections parallel to the plane  $yz$  are hyperbolas. For  $y = 0$ ,  $z = \pm mx$ , whose locus is two intersecting right lines. Hence, *any plane section of a cone parallel to its axis is a hyperbola, and any section containing the axis is two intersecting right lines.*

A **Conic Section** is the section of a cone made by a plane.

**245.** To determine the nature of any conic section that is not parallel to the axis of the cone, we find the equation of any such section referred to axes in its own plane.

Let  $HPN$  be any section of the cone  $VDY'N$  passing through the axis of  $y$ ; then this section will be perpendicular to the plane  $xz$ . The cone, and therefore the section, is symmetrical with respect to the plane  $xz$ . Refer this section to  $ON$

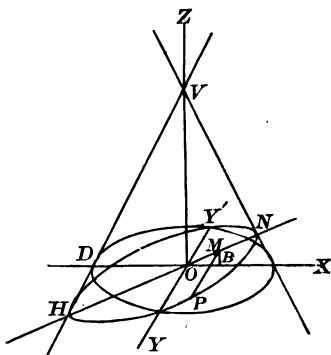


Fig. 102.

and  $OY$  as the axes of  $x$  and  $y$  respectively. Let  $(x, y, z)$  be the point  $P$  referred to the co-ordinate planes, and  $(x', y')$  be  $P$  referred to  $ON$  and  $OY$ . Let  $XON = \phi$  and  $ODV = \theta$ . Draw  $PM$  perpendicular to  $ON$ ; then it will be perpendicular to plane  $xz$ , and we have

$$y = y', \quad OB = OM \cos \phi, \quad \text{or } x = x' \cos \phi;$$

$$BM = OM \sin \phi, \quad \text{or } z = x' \sin \phi.$$

Substituting these values of  $x, y, z$  in [73], we obtain

$$\tan^2 \theta (x'^2 \cos^2 \phi + y'^2) = (x' \sin \phi - c)^2.$$

Omitting accents and performing indicated operations, we have

$$y^2 \tan^2 \theta + x^2 (\cos^2 \phi \tan^2 \theta - \sin^2 \phi) + 2cx \sin \phi - c^2 = 0.$$

Substituting  $\cos^2 \phi \tan^2 \theta$  for  $\sin^2 \phi$ , we obtain

$$y^2 \tan^2 \theta + x^2 \cos^2 \phi (\tan^2 \theta - \tan^2 \phi) + 2cx \sin \phi - c^2 = 0, \quad [74]$$

which is the equation of the conic  $NPH$  referred to  $ON$  and  $OY$  as axes.

By giving to  $c$  all values between 0 and  $\infty$ , and to  $\phi$  all values between  $0^\circ$  and  $90^\circ$ , equation [74] is made to represent any section of a cone except those parallel to its axis, which have already been considered.

Discussion of equation [74].

Here  $\Sigma = 4 \cos^2 \phi \tan^2 \theta (\tan^2 \theta - \tan^2 \phi)$ ,

$\Delta = 4c^2 [\cos^2 \phi \tan^2 \theta (\tan^2 \theta - \tan^2 \phi) + \tan^2 \theta \sin^2 \phi]$ .

(i.) *First suppose  $c$  not equal to zero.*

Let  $\phi < \theta$ ; then  $\tan^2 \phi < \tan^2 \theta$ ,  $\Sigma$  is positive, and  $\Delta$  is not zero; hence the section is an ellipse.

Let  $\phi = \theta$ ; then  $\tan^2 \phi = \tan^2 \theta$ ,  $\Sigma = 0$ , and  $\Delta$  is not zero; hence the section is a parabola.

Let  $\phi > \theta$ ; then  $\tan^2 \phi > \tan^2 \theta$ ,  $\Sigma$  is negative, and  $\Delta$  is not zero; hence the section is a hyperbola.

Hence, when the cutting plane does not pass through the vertex of the cone, *the section is an ellipse, a parabola, or a hyperbola, according as the angle which the cutting plane makes with the base of the cone is less than, equal to, or greater than that made by an element.*

(ii.) If  $c = 0$ ,  $\Delta = 0$ ; hence, when the cutting plane passes through the vertex, the elliptical section reduces to a point, the parabolic to a straight line, and the hyperbolic to two intersecting right lines.

If  $\phi = 0$ , the cutting plane is perpendicular to the axis of the cone, and equation [74] becomes

$$y^2 + x^2 = c^2 \cot^2 \theta,$$

whose locus is a circle.

If  $c = \infty$ , the cone becomes a cylinder, and the section made by a plane parallel to an element is two parallel lines or a single right line.

## Ex. 46.

1. What is the locus in space of  $x^2 + 3x^2 - 6x - 8 = 0$ ? of  $y^2 - 2y^2 - 5y + 6 = 0$ ? of  $z^2 + mz = 0$ ?

2. What is the locus in space of  $y^2 = 8x$ ? of  $4x^2 + 9y^2 = 36$ ? of  $9z^2 - 16y^2 = 144$ ? of  $z^2 + x^2 = r^2$ ? of  $(2a - z)(y^2 - b^2) = 0$ ?

3. Find the equations of the projecting cylinders of the curve  $x^2 + 3y^2 - 2z^2 = 8$ ,  $x^2 + 2y^2 + 3z^2 = 16$ .

4. Find the equations of the projections of the curves

$$x^2 + y^2 + 2z^2 = 16, \quad 9(x^2 + y^2) + 4z^2 = 36.$$

5. Find the semi-axes and eccentricity of the ellipse

$$4x^2 + 9y^2 + 4z^2 = 37, \quad z = \frac{1}{2}.$$

6. Find the nature of the curves

$$x^2 + y^2 + 4z^2 = 25, \quad 7(x^2 + y^2) - 4z^2 = 79.$$

7. Find the traces of the surface  $2x^2 + 5y^2 - 7z^2 = 9$ ; of the surface  $x^2 + 3y^2 = 8z$ .

8. Find the equation of the surface of revolution whose axis is the axis of  $z$ , and one of whose traces is  $z = \pm 3x + 5$ ; find its trace on the plane  $xy$ .

9. Find the equation of a cone of revolution one of whose traces is  $x^2 + y^2 = 9$ , and whose vertex is  $(0, 0, 5)$ .

10. Find the equation of the paraboloid of revolution one of whose traces is  $2x^2 = 3z + 5$ .

11. Find the equation of the paraboloid of revolution one of whose traces is  $y^2 = 8x$ .

12. Find the equation of the cone of revolution whose axis is the axis of  $z$ , and one of whose traces is  $2y = \pm z + 6$ ; find its vertex.

13. Find the equation of the surface of revolution whose axis is the axis of  $z$ , and one of whose traces is  $9x^2 + 4z^2 = 36$ .

14. Find the equation of the surface of revolution whose axis is the axis of  $z$ , and one of whose traces is  $16y^2 + 9z^2 = 144$ .

15. Find the equation of the surface of revolution whose axis is the axis of  $z$ , and one of whose traces is  $9z^2 - 4y^2 = -36$ .

16. Find the equation of the surface of revolution whose axis is the axis of  $z$ , and one of whose traces is  $z^2x = 1$ ; also when one trace is  $z^2 = 2y^2$ .

17. Each element of a cone makes an angle of  $45^\circ$  with its axis; find the semi-axes of the section made by a plane cutting the axis 5 below the vertex and at an angle of  $60^\circ$ .

### SUPPLEMENTARY PROPOSITIONS.

**246.** *To find the general equation of the sphere.*

Let  $r$  denote the radius of any sphere,  $(a, b, c)$  its centre, and  $(x, y, z)$  any point on its surface. Then, since  $r$  is the distance between the points  $(a, b, c)$  and  $(x, y, z)$ , we have

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2, \quad [75]$$

or  $x^2 + y^2 + z^2 - 2ax - 2by - 2cz = r^2 - a^2 - b^2 - c^2$ , (1)  
which is the general rectangular equation of the sphere.

If the origin be at the centre, then  $a = b = c = 0$ , and [75] becomes  $x^2 + y^2 + z^2 = r^2$ . (2)

From (1) it follows that any equation of the form

$$x^2 + y^2 + z^2 + Gx + Hy + Iz = K \quad (3)$$

is the equation of a sphere.

Any equation of the form of (3) can readily be reduced to the form of [75], from which the centre and radius of its locus become known.

Since (3) or [75] contains four arbitrary constants, a sphere may in general be made to pass through any four given points.

**247.** *The intersection of two spheres is a circle.*

Let the equations of the two spheres be

$$x^2 + y^2 + z^2 + Gx + Hy + Iz = K, \quad (1)$$

$$\text{and} \quad x^2 + y^2 + z^2 + G'x + H'y + I'z = K'. \quad (2)$$

Subtracting (2) from (1), we obtain

$$(G - G')x + (H - H')y + (I - I')z = K - K'. \quad (3)$$

Hence the intersection of the spheres (1) and (2) lies in the plane (3), and is the same as the intersection of (1) and (3). But the plane section of a sphere is a circle. Hence the intersection of the two spheres is a circle.

**248.** *To find the equation of the tangent plane to a sphere at a given point.*

Let the given point be  $(x_1, y_1, z_1)$ ; then the equation of the radius to this point, that is, of the line passing through  $(a, b, c)$  and  $(x_1, y_1, z_1)$ , is

$$\frac{x - x_1}{a - x_1} = \frac{y - y_1}{b - y_1} = \frac{z - z_1}{c - z_1}. \quad (1)$$

Now the tangent plane is perpendicular to (1) at the point  $(x_1, y_1, z_1)$ ; but the equation of the plane through  $(x_1, y_1, z_1)$  perpendicular to (1) is

$$(a - x_1)(x - x_1) + (b - y_1)(y - y_1) + (c - z_1)(z - z_1) = 0, \quad (2)$$

which is, therefore, the equation of the tangent plane.

If the origin is at the centre,  $a = b = c = 0$ , and (2) becomes

$$xx_1 + yy_1 + zz_1 = r^2.$$

## TRANSFORMATION OF CO-ORDINATES.

**249.** *To change the origin of co-ordinates without changing the direction of the axes.*

Let  $(m, n, q)$  be the new origin referred to the old axes. Let  $x, y, z$  be the old, and  $x', y', z'$  the new co-ordinates of any point  $P$ ; then, evidently, we have

$$x = m + x', y = n + y', z = q + z'.$$

Hence, to find the equation of a locus referred to new parallel axes whose origin is  $(m, n, q)$ , substitute  $m + x, n + y$ , and  $q + z$ , for  $x, y$ , and  $z$ , respectively.

**250.** *To change the direction of the axes without changing the origin.*

Let  $\alpha_1, \beta_1, \gamma_1; \alpha_2, \beta_2, \gamma_2; \alpha_3, \beta_3, \gamma_3$  be, respectively, the direction angles of the new axes  $OX', OY', OZ'$ , referred to the

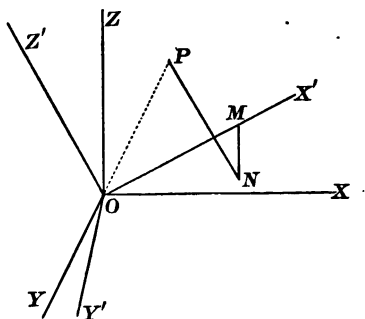


Fig. 103.

old axes  $OX, OY, OZ$ . Let  $x, y, z$  be the old, and  $x', y', z'$  the new co-ordinates of any point  $P$ . Draw  $PN$  perpendicular to the plane  $X'OY'$ , and  $NM$  perpendicular to  $OX'$ ; then  $OM = x', MN = y'$ , and  $NP = z'$ . Now the projection of

$OP$  on  $OX$  ( $=x$ ) is equal to the sum of the projections of  $OM$ ,  $MN$ , and  $NP$  on the same line; hence

$$x = x' \cos \alpha_1 + y' \cos \alpha_2 + z' \cos \alpha_3. \quad (1)$$

In like manner, we obtain

$$y = x' \cos \beta_1 + y' \cos \beta_2 + z' \cos \beta_3, \quad (2)$$

and 
$$z = x' \cos \gamma_1 + y' \cos \gamma_2 + z' \cos \gamma_3. \quad (3)$$

Hence, to change the direction of the axes without changing the origin, *substitute for  $x$ ,  $y$ , and  $z$ , their values as given in equations (1), (2), and (3).*

Since the values of  $x$ ,  $y$ ,  $z$  are each of the first degree in  $x'$ ,  $y'$ ,  $z'$ , any transformation of co-ordinates cannot change the degree of an equation. (§ 91.)

**251. Quadrics.** The locus of an equation of the second degree between three variables is called a **Quadric**. Thus the general equation of a quadric is

$$\begin{aligned} Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy \\ + Iz + K = 0. \end{aligned} \quad (1)$$

Putting  $z = q$  in (1), we obtain

$$\begin{aligned} Ax^2 + Dxy + By^2 + (Eq + G)x + (Fq + H)y \\ + (Cq^2 + Iq + K) = 0. \end{aligned} \quad (2)$$

Since the locus of (2) in the plane  $xy$  is a conic, and since the coefficients  $A$ ,  $D$ ,  $B$  are the same for all values of  $q$ , all plane sections of the quadric (1), parallel to the plane  $xy$ , are similar conics. Now the axis of co-ordinates may be so changed that the new plane  $xy$  will be one of any system of parallel planes cutting the quadric. But, as this transformation does not change the degree of the equation, it follows that

*All parallel plane sections of any quadric are similar conics.*



**252.** By transformations of co-ordinates the general equation (1) of § 251 may be reduced to one of the two following simple forms: \*

$$Px^2 + Qy^2 + Rz^2 = S. \quad (1)$$

$$Px^2 + Qy^2 = Uz. \quad (2)$$

Now whatever be the values or signs of  $P, Q, R, S$ , equation (1) evidently represents central quadrics. But the loci of (2) have no centre; for if they had, and the origin were changed to that centre, the first power of  $z$  would disappear from the equation. But no expression of the form  $q + z$ , when substituted for  $z$ , can cause  $z$  to disappear.

Hence (2) represents non-central quadrics.

**253. Central Quadrics.** If neither  $P, Q, R$ , nor  $S$  be zero in (1) of § 252, we have

$$\frac{x^2}{S \div P} + \frac{y^2}{S \div Q} + \frac{z^2}{S \div R} = 1,$$

which can be written in the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad (A)$$

\* By changing the direction of the axes the general equation can in all cases be reduced to the form,

$$Px^2 + Qy^2 + Rz^2 + G'x + H'y + I'z - K = 0. \quad (1)$$

This transformation is analogous to that in § 189.

(i.) If no one of the three coefficients  $P, Q, R$  be zero, by a change of origin, as in § 188, we obtain

$$Px^2 + Qy^2 + Rz^2 = S. \quad (2)$$

(ii.) If any one of these coefficients be zero, for example  $R$ , by a change of origin, we obtain

$$Px^2 + Qy^2 = Uz. \quad (3)$$

If two of these coefficients are zero, (1) can be reduced to a form embraced in (3) by first changing the origin and then the direction of the axes.

$$\text{or} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \quad (\text{B})$$

$$\text{or} \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \quad (\text{C})$$

according as  $S \div P$ ,  $S \div Q$ ,  $S \div R$  are all positive, two positive and one negative, or one positive and two negative. [If all three are negative there is no real locus.]

If  $S$  be zero, we have

$$Px^2 + Qy^2 + Rz^2 = 0. \quad (\text{D})$$

If either  $P$ ,  $Q$ , or  $R$  be zero in (1), its locus is a cylindrical surface by (ii.) of § 236.

**254.** A discussion of (A) discovers the following properties of its locus.

- (i.) Its traces on each of the co-ordinate planes are ellipses.
- (ii.) All plane sections parallel to either co-ordinate plane are similar ellipses.
- (iii.) The quadric is included between the tangent planes

$$x = \pm a, y = \pm b, z = \pm c.$$

The quadric (A) is called an **Ellipsoid**. If  $a = b$ , the ellipsoid is the oblate or prolate spheroid, according as  $a >$  or  $<$   $c$ .

The ellipsoid may evidently be generated by a variable ellipse moving parallel to the plane  $xy$  with its centre in the axis of  $z$ , its axes being chords of the traces of the quadric on the planes  $yz$  and  $zx$ .

**255.** A discussion of (B) discovers the following properties:

- (i.) Its trace on the plane  $xy$  is an ellipse, while its traces on the planes  $yz$  and  $zx$  are hyperbolas, whose transverse axes lie on the axes of  $y$  and  $x$  respectively.

- (ii.) All plane sections parallel to the plane  $xy$  are ellipses, while all plane sections parallel to the plane  $yz$  and  $zx$  are

hyperbolas. The smallest elliptical section is the trace on the plane  $xy$ . The semi-axes of this ellipse are  $a$  and  $b$ .

The locus of (B) is called the **Hyperboloid of One Nappe**.

If  $a = b$ , the locus of (B) is a hyperboloid of revolution.

The hyperboloid can evidently be traced by a variable ellipse parallel to the plane  $xy$ , whose centre moves along the axis of  $z$ , and whose axes are the chords of the traces of the quadric on the planes  $yz$  and  $zx$ .

**256.** From a discussion of (C), we learn that

(i.) Its traces on the planes  $yx$  and  $zx$  are hyperbolas whose transverse axes are on the axis of  $x$ .

(ii.) The plane sections parallel to the plane  $zy$  are ellipses, and no portion of the quadric lies between the tangent planes  $x = \pm a$ .

(iii.) The plane sections parallel to the planes  $yx$  and  $zx$  are hyperbolas whose transverse axes are parallel to the axis of  $x$ .

The locus of (C) is called the **Hyperboloid of Two Nappes**.

**257.** If the coefficients of (D) are all positive or all negative, its locus is the point  $(0, 0, 0)$ . If two coefficients are negative and one positive, by dividing by  $-1$ , two become positive and one negative. Hence we need discuss only the form represented by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0, \quad (D')$$

from which we learn that

(i.) All plane sections parallel to the planes  $yz$  and  $zx$  are hyperbolas whose transverse axes are parallel to the axis of  $z$ .

(ii.) All plane sections parallel to the plane  $xy$  are ellipses, the trace on this plane being a point.

(iii.) The traces on the planes  $yz$  and  $zx$  are each two right lines intersecting at the origin.

(iv.) All plane sections through the axis of  $z$  are two right lines intersecting at the origin.

For denote any plane through the axis of  $z$  by

$$y = mx. \quad (1)$$

Eliminating  $y$  between (1) and  $(D')$ , we obtain

$$z = \pm \frac{cx}{ab} \sqrt{b^2 + a^2 m^2}. \quad (2)$$

Now the intersections of (1) and  $(D')$  are the same as those of (1) and (2), which are evidently two right lines passing through the origin.

Hence the locus of  $(D')$  is a cone whose axis is the axis of  $z$ , and whose directrix is an ellipse. If  $a = b$ , it becomes a cone of revolution.

**258. Non-Central Quadrics.** If neither  $P$ ,  $Q$ , nor  $U$  be zero in (2) of § 252, we have

$$\frac{x^2}{U + P} + \frac{y^2}{U + Q} = z,$$

which can be written in the form

$$\frac{x^2}{l} + \frac{y^2}{l'} = z, \quad (E)$$

$$\text{or} \quad \frac{x^2}{l} - \frac{y^2}{l''} = z, \quad (F)$$

according as  $P$  and  $Q$  have like or unlike signs.

A discussion of (E) discovers the following properties :

(i.) Plane sections parallel to the plane  $xy$  are ellipses, and the surface lies above the tangent plane  $z = 0$ .

(ii.) All plane sections parallel to the plane  $yz$  or  $xz$  are parabolas, and the traces on these planes are parabolas, having the axis of  $z$  as their common axis and their concavities upward.

By a discussion of (F) we learn that

(i.) The traces on the planes  $yz$  and  $zx$  are parabolas whose axes lie on the axis of  $z$ , and whose concavities are in opposite directions.

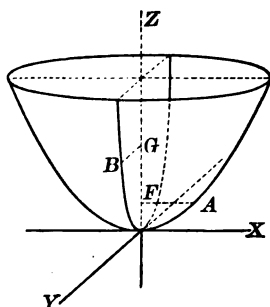
(ii.) Plane sections parallel to the planes  $yz$  and  $zx$  are parabolas whose concavities are in opposite directions.

(iii.) Plane sections parallel to the plane  $xy$  are hyperbolas whose transverse axes are parallel to the axis of  $x$ , or  $y$ , according as  $z$  is positive or negative. The trace on this plane is two intersecting right lines.

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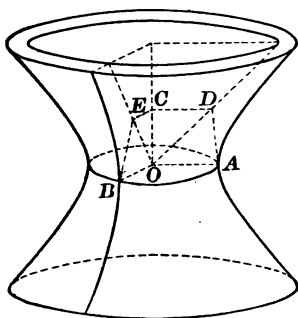
### DIAGRAMS.

NOTE. These figures are taken, by permission, from W. B. Smith's Geometry.



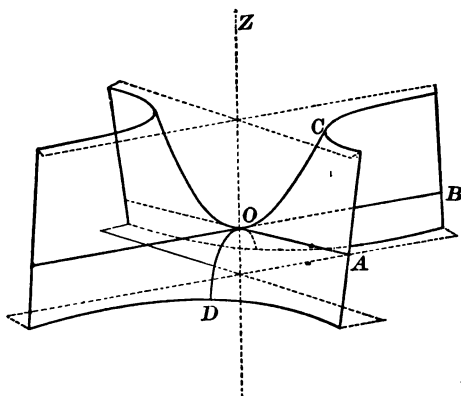
ELLIPTIC PARABOLOID.

$FA = 2a$  and  $GB = 2b$  are  
half-parameters.



SIMPLE HYPERBOLOID.

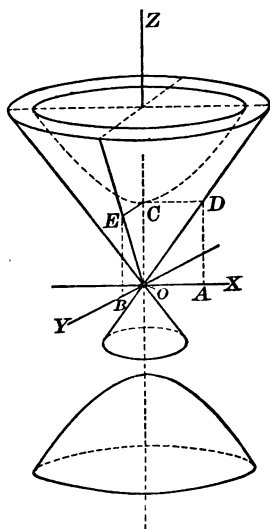
$AB$  is Ellipse of the Gorge.  
 $EOD$  is the Asymptotic Cone.



HYPERBOLIC PARABOLOID.

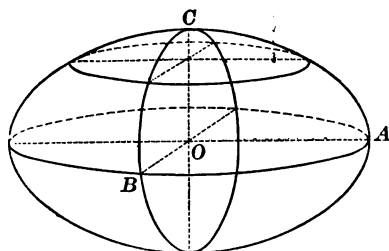
$OC$  and  $OD$  are Parabolas.

$OA$  and  $OB$  are Asymptotic directions for the Hyperbolas.



DOUBLE HYPERBOLOID.

$EOD$  is the Asymptotic Cone.



ELLIPSOID.

$OA = a$ ,  $OB = b$ ,  $OC = c$ .



# ANSWERS.



## Ex. 3. Page 7.

1. Let  $x_1 = -2$ ,  $y_1 = 5$ ,  $x_2 = -8$ ,  $y_2 = -3$ . Substituting in [1], we have

$$d = \sqrt{(-6)^2 + (-8)^2} = \sqrt{100} = 10.$$

In Fig. 3 the points  $P$  and  $Q$  are plotted to represent this case. If we choose to solve the question without the aid of [1], we may neglect algebraic signs, and we have

$$QR = NO - MO = 8 - 2 = 6;$$

$$PR = PM + MR = 5 + 3 = 8;$$

$$\therefore PQ^2 = QR^2 + PR^2 = 36 + 64 = 100, \text{ and } PQ = 10.$$

2. 13.

8. 5, 5, 6.

3. 5.

9.  $a$ ,  $b$ ,  $\sqrt{a^2 + b^2}$ .

4. 10.

10.  $\sqrt{29}$ , 5,  $2\sqrt{10}$ ,  $4\sqrt{5}$ ;

5.  $2\sqrt{a^2 + b^2}$ .

$2\sqrt{10}$ ,  $\sqrt{117}$ .

6. 25, 29,  $20\sqrt{2}$ .

11. 8 or -16.

7.  $2\sqrt{17}$ ,  $5\sqrt{2}$ ,  $\sqrt{106}$ .

12.  $(x-7)^2 + (y+2)^2 = 121$ .

13.  $(x-2)^2 + (y-3)^2 = (x-4)^2 + (y-5)^2$ , which reduces to  $x+y=7$ .

## Ex. 4. Page 9.

1. (6, 6).

3. (2, -2).

5. (7, 1).

2. (-1, 0).

4.  $(3, -1)$ ,  $(\frac{1}{2}, \frac{1}{2})$ ,  $(-\frac{1}{2}, -\frac{1}{2})$ .

6.  $(a, -b)$ .

7. Take the origin of co-ordinates at the intersection of the two legs, and the axes of  $x$  and  $y$  in the directions of the legs. Then, if  $a$  and  $b$  denote the lengths of the legs, the co-ordinates of the three vertices will be (0, 0),  $(a, 0)$ , and  $(0, b)$ .

10. Observe that now the distances  $RB$  and  $BQ$  will be  $x-x_2$  and  $y-y_2$ .

12.  $(\frac{2}{3}, \frac{1}{3})$ .

14.  $(7\frac{1}{2}, -31\frac{1}{2})$ .

11. (6, 2).

13. (8, 0).

15. (13, -1), (-11, 5), (1, -11).



4

## Ex. 7. Page 23.

- |   |  |
|---|--|
| 1. 12, 16.  | 14. Locus does not cut the axes.       |
| 2. -10, 6.  | 15. (5, 7).                            |
| 3. $\pm 4, \pm 4$ .   | 16. (2, 1).                            |
| 4. $\pm \frac{4}{3}, \pm 2$ .   | 17. (3, <del>4</del> ) and (-4, 3).    |
| 5. $\pm \frac{4}{3}$ , imaginary.   | 18. (3, 4).                            |
| 6. $\pm \frac{4}{3}, -4$ .  | 19. (5, 3) and (3, 5).                 |
| 7. $\pm b, \pm a$ .   | 20. (0, 0) and (2, 4).                 |
| 8. 3 on $OX$ .  | 21. (5, -3), (6, 4), (-4, -1).         |
| 9. $\pm 3$ on $OX$ .  | 22. $\sqrt{61}$ , 5, $\sqrt{104}$ .    |
| 10. Locus passes through origin.  | 23. 3, 4, 5.                           |
| 11. Locus passes through origin.  | 24. $(a, b)(-a, b), (-a, -b)(a, -b)$ . |
| 12. $\begin{cases} \text{On } OX, 8, \text{ and } -4. \\ \text{On } OY, 4 \pm 4\sqrt{3}. \end{cases}$ | 25. No.                                |
| 13. Locus passes through origin.  | 26. 10.                                |

## Ex. 9. Page 31.

1. Let  $x$  and  $y$  denote the variable co-ordinates of the moving point. Then it is evident that for all positions of the point  $y = 3x$ . Therefore the required equation is  $y = 3x$  or  $y - 3x = 0$ . Does the locus of this equation pass through the origin?

2.  $x - 6 = 0, x + 6 = 0, x = 0$ .

3.  $y - 4 = 0, y + 1 = 0, y = 0$ .

4. The line  $x = 3$  is the line  $AB$  (Fig. 74); how is this line drawn? The locus of the variable point consists of the two parallels to  $AB$ , drawn at the distance 2 from  $AB$ . Let  $CD, EF$ , be these parallels, and  $(x, y)$  denote in general the variable point, then for all points in  $CD$   $x = 3 + 2 = 5$ , and for all points in  $EF$   $x = 3 - 2 = 1$ . Therefore the equation of the line  $CD$  is  $x - 5 = 0$ , and that of the line  $EF$  is  $x - 1 = 0$ . The product of these two equations is the equation  $(x - 5)(x - 1) = 0$ . This equation is evidently satisfied by every point in each of the lines  $CD$  and  $EF$ , and by no other points. Therefore the required equation is  $(x - 5)(x - 1) = 0$ , or  $x^2 - 6x + 5 = 0$ . Verify that this equation is satisfied by points taken at random in the lines  $CD$  and  $EF$ .

5.  $y^2 - 10y + 16 = 0$ , two parallel lines.

6.  $x^2 + 8x - 9 = 0$ , two parallel lines.      7.  $x + 3 = 0$ ,  $y - 2 = 0$ .

8. It is proved in elementary geometry that all points equidistant from two given points lie in the perpendicular erected at the middle point of the line joining the two given points. This perpendicular is the locus required, and its equation evidently is  $x = 3$ .

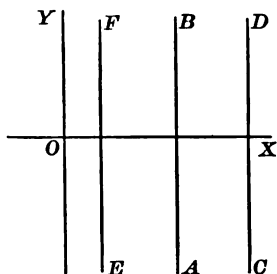


Fig. 74.

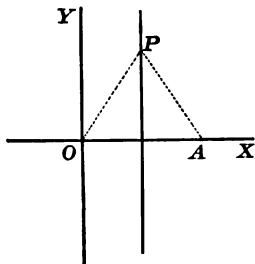


Fig. 75.

Let us now solve this problem by the analytic method. Let  $O$  (Fig. 75) be the origin,  $A$  the point  $(6, 0)$ , and let  $P$  represent any position of a point equidistant from  $O$  and  $A$ ,  $x$  and  $y$  its two co-ordinates. Then from the given condition

$$PO = PA.$$

Therefore  $x^2 + y^2 = (x - 6)^2 + (y - 0)^2$ ,  
 or  $x^2 + y^2 = x^2 - 12x + 36 + y^2$ ;  
 whence  $x = 3$ ,

the equation of the locus required.

9.  $x - 1 = 0$ .    10.  $y - 2 = 0$ .    11.  $x - 3y - 1 = 0$ .    12.  $x - y = 0$ .

13.  $x^2 + y^2 = 100$ , a circle with the origin for centre and 10 for radius.

14. Express by an equation the fact that the distance from the point  $(x, y)$  to the point  $(4, -3)$  is equal to 5. The equation is  $(x - 4)^2 + (y + 3)^2 = 25$ .

15.  $(x + 4)^2 + (y + 7)^2 = 64$ .      16.  $x^2 + y^2 = 81$ .

17. Draw  $AO \perp$  to  $BC$  (Fig. 76). Take  $AO$  for the axis of  $x$ , and  $BC$  for the axis of  $y$ ; then  $A$  is the point  $(3, 0)$ .

Let  $P$  represent any position of the vessel,  $x$  and  $y$  its co-ordinates

$OM$  and  $PM$ . Join  $PA$ , and draw  $PQ \perp BC$ , and meeting it in  $Q$ . Then from the given condition

$$PA = PQ = OM.$$

Therefore

$$PA^2 = OM^2.$$

Now  $PA^2 = AM^2 + PM^2 = (x-3)^2 + y^2$ , and  $OM^2 = x^2$ . Substituting, we have

$$(x-3)^2 + y^2 = x^2;$$

whence

$$y^2 = 6x - 9.$$

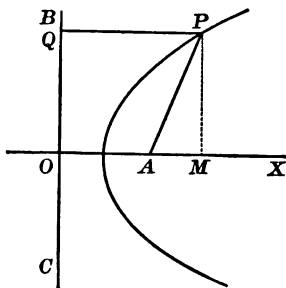


Fig. 76.

The locus is the curve called the parabola. We leave the discussion of the equation as an exercise for the learner.

18. If  $BC$  is taken for the axis of  $y$ , and the perpendicular from  $A$  to  $BC$  as the axis of  $x$ , the required equation is  $y^2 = 12x - 36$ .

19.  $x^2 - 3y^2 = 0$ , two straight lines.

20.  $x^2 + y^2 = k^2 - a^2$ , a circle.

21.  $4ax \pm k^2 = 0$ , two straight lines.



### Ex. 10. Page 33.

4.  $d = \sqrt{x_1^2 + y_1^2}$ .

6.  $x + y = 7$ .

5.  $(x-4)^2 + (y-6)^2 = 64$ .

7.  $(\frac{1}{3}, \frac{2}{3}); \frac{5}{3}\sqrt{2}$ .

8. Take two sides of the rectangle for the axes, and let  $a$  and  $b$  represent their lengths; then the vertices of the rectangle will be the points  $(0, 0)$ ,  $(a, 0)$ ,  $(a, b)$ ,  $(0, b)$ .

9. Take one vertex as the origin, and one side,  $a$ , as the axis of  $x$ ; then  $(0, 0)$  and  $(a, 0)$  will be two vertices. Let  $(b, c)$  be a third vertex; then  $(a+b, c)$  will represent the fourth.

10.  $(11, 2), (-1, 4), (15, 16)$ . 11.  $(5, -2), (\frac{1}{2}, +\frac{1}{2}), (\frac{3}{2}, -\frac{1}{2})$ .  
 12.  $(1, -\frac{1}{3})$ . 13.  $\sqrt{17}$ . 14.  $(\frac{1}{2}, \frac{1}{2})$ . 16.  $(6, 23)$ .  
 17.  $(\frac{x_1+3x_2}{4}, \frac{y_1+3y_2}{4}), (\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}), (\frac{3x_1+x_2}{4}, \frac{3y_1+y_2}{4})$ .  
 21. 3 or -23. 23.  $(8, 6)$  and  $(8, -6)$ .  
 22.  $\begin{cases} 3 \text{ and } 2 \text{ on } OX. \\ 6 \text{ and } 1 \text{ on } OY. \end{cases}$  24.  $(2a, a)$  and  $(-2a, a)$ .  
 25.  $(a, 0)$  and  $(-a, 0)$ .  
 26.  $10, \sqrt{104}, \sqrt{52}$ .  
 27. Taking the fixed lines for axes, the equation is  $y = 6x$  or  $x = 6y$ .  
 28. Taking  $A$  for origin, and  $AB$  for the axis of  $x$ , the equation is  $x^2 - 3y^2 = 0$ .  
 29. Taking the fixed line and the perpendicular to it from the fixed point as the axes of  $x$  and  $y$  respectively, the required equation is  $x^2 + (y - a)^2 = 4y^2$ .

## Ex. 11. Page 40.

- |  |                                       |
|--|---------------------------------------|
| 1. $x - y + 1 = 0$ .                         | 20. $y + 3 = 0$ .                     |
| 2. $2x - y - 3 = 0$ .                        | 21. $x - 2 = 0$ .                     |
| 3. $x + y - 1 = 0$ .                         | 22. $x - y + 2 = 0$ .                 |
| 4. $x - y = 0$ .                             | 23. $x - y + 5 = 0$ .                 |
| 5. $3x + 2y - 12 = 0$ .                      | 24. $x - y - 4 = 0$ .                 |
| 6. $2x - 3y + 6 = 0$ .                       | 25. $x - \sqrt{3}y - 4\sqrt{3} = 0$ . |
| 7. $x + y - 7 = 0$ .                         | 26. $y + 4 = 0$ .                     |
| 8. $4x - 3y = 0$ .                           | 27. $\sqrt{3}x - y - 4 = 0$ .         |
| 9. $y = 0$ .                                 | 28. $x = 0$ .                         |
| 10. $y = 4$ .                                | 29. $\sqrt{3}x + y + 4 = 0$ .         |
| 11. $5x - 2y = 0$ .                          | 30. $x + y + 4 = 0$ .                 |
| 12. $nx - my = 0$ .                          | 31. $x + \sqrt{3}y + 4\sqrt{3} = 0$ . |
| 13. $x - y - 3 = 0$ .                        | 32. $y + 4 = 0$ .                     |
| 14. $\sqrt{3}x - y + 7 - 2\sqrt{3} = 0$ .    | 33. $3x + 4y - 12 = 0$ .              |
| 15. $x - y + 14 = 0$ .                       | 34. $x - 3y + 6 = 0$ .                |
| 16. $\sqrt{3}x + 3y + 12 - 13\sqrt{3} = 0$ . | 35. $x + y + 3 = 0$ .                 |
| 17. $\sqrt{3}x - 3y - 3\sqrt{3} = 0$ .       | 36. $3x - 5y - 15 = 0$ .              |
| 18. $x + y - 3 = 0$ .                        | 37. $x - 2y + 10 = 0$ .               |
| 19. $\sqrt{3}x + y = 0$ .                    | 38. $x - y - 1 = 0$ .                 |

39.  $x - y - n = 0$ .

40.  $4x + y - 4n = 0$ .

41.  $x + y - 5\sqrt{2} = 0$ .

42.  $x - y\sqrt{3} + 10 = 0$ .

43.  $x + y\sqrt{3} + 10 = 0$ .

44.  $x - y\sqrt{3} - 10 = 0$ .

45.  $\begin{cases} x + 7y + 11 = 0, & x - 3y + 1 = 0, \\ 3x + y - 7 = 0. \end{cases}$

46.  $\begin{cases} x - 7y = 39, & 9x - 5y = 3, \\ 4x + y = 11. \end{cases}$

47.  $\begin{cases} 17x - 3y = 25, & 7x + 9y \\ = -17, & 5x - 6y - 21 = 0. \end{cases}$

48.  $\begin{cases} 5x - y = 0, & 5x + 6y - 35 = 0, \\ 3x - y = 21, & 9x + 4y = 0, \\ y = 0, & 14x + 3y = 29. \end{cases}$

49.  $x - y\sqrt{3} - 7\frac{1}{2} = 0$ .

50.  $y = x + 3$ .

51.  $x + y + 6\sqrt{2} = 0$ .

52.  $x + y + 6\sqrt{2} = 0$ .

53.  $\frac{x}{-\frac{11}{3}} + \frac{y}{\frac{11}{2}} = 1$ .

55.  $a = \frac{C}{A}$ , or  $-\frac{b}{m}$ ,  $b = \frac{C}{B}$ .

58.  $m = -\frac{A}{B}$ , or  $-\frac{b}{a}$ ,  $b = \frac{C}{B}$ .

59.  $(5, -3), (6, 4), (-4, -1)$ .

60.  $9x + 2y = 0, \frac{1}{8}\sqrt{85}$ .

61.  $y \pm x = y_1 \pm x_1$ .

62.  $\begin{cases} (d-c)x - (b-a)y = ad - bc, \\ (d-c)x + (b-a)y = bd - ac. \end{cases}$

64.  $\begin{cases} 2y_2x + (x_1 - 2x_2)y - x_1y_2 = 0, \\ y_2x + (2x_1 - x_2)y - x_1y_2 = 0, \\ y_2x - (x_1 + x_2)y = 0. \end{cases}$

65.  $m = 4$ .

66.  $m = 3$ .

67.  $b = -9$ .

68.  $\frac{y_3 - y_1}{x_3 - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$ , or,  $x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0$ .

## Ex. 12. Page 44.

1.  $-\frac{2}{13}\sqrt{13}x + \frac{2}{13}\sqrt{13}y = \frac{11}{13}\sqrt{13}$ ;  $p = \frac{11}{13}\sqrt{13}$ .

2.  $\frac{3}{34}\sqrt{34}x + \frac{3}{34}\sqrt{34}y = \frac{13}{34}\sqrt{34}$ ;  $p = \frac{13}{34}\sqrt{34}$ .

3.  $p = \frac{1}{17}\sqrt{17}$ .

4.  $p = \frac{1}{13}\sqrt{13}$ .

5.  $p = \frac{1}{2}\sqrt{26}$ .

6.  $p = \frac{1}{10}\sqrt{50}$ .

7.  $p = \frac{n}{\sqrt{e^2 + c^2}}$ .

8.  $p = \frac{r}{\sqrt{n^2 + c^2}}$ .

9. Fourth quadrant.

10. Second quadrant.

11. Fourth quadrant.

12. Second quadrant.

13. Third quadrant.

14. First quadrant.

15. Second quadrant.

16. Fourth quadrant.

17. Third quadrant. 20.  $m = -\frac{b}{a}$ .
18. Fourth quadrant. 21. 0; 8.
22. If  $C = 12$ ,  $A = 4$ ,  $B = -1$ .
24.  $A = (y_2 - y_1)$ ,  $B = -(x_2 - x_1)$ ,  $C = (x_1 y_2 - x_2 y_1)$ .
25.  $m = \frac{y_2 - y_1}{x_2 - x_1}$ ,  $b = \frac{x_2 y_1 - x_1 y_2}{x_2 - x_1}$ .

## Ex. 13. Page 46.

1.  $3x - y - 16 = 0$ . 5.  $x - 5 = 0$ .
2.  $3x - 4y - 3 = 0$ . 6.  $x + 4y + 49 = 0$ .
3.  $4x - y = 0$ . 7.  $7x - 23y + 193 = 0$ .
4.  $y - 8 = 0$ . 8.  $y = 2x$ .
9.  $35y + 49x - 79 = 0$ .

## Ex. 14. Page 47.

2.  $\tan \phi = \frac{1}{2}$ . 3.  $\tan \phi = \frac{1}{18}$ . 4.  $\tan \phi = \frac{n}{n^2 + 2}$ .
5.  $90^\circ$ . 6.  $135^\circ$ . 7.  $90^\circ$ . 8.  $0^\circ$ . 9.  $30^\circ$ .
11.  $\begin{cases} y = 5x - 10, \\ x + 5y = 28. \end{cases}$  13.  $\begin{cases} y - 3 = m'(x - 2), \\ \text{and } m' = -(8 \pm 5\sqrt{3}). \end{cases}$
12.  $\begin{cases} y = 5x + 11, \\ x + 5y - 3 = 0. \end{cases}$  14.  $\begin{cases} y - 3 = m'(x - 1), \\ \text{and } m' = \frac{8 \pm 5\sqrt{3}}{11}. \end{cases}$
22.  $2x + 3y - 31 = 0$ .
23.  $62x + 31y - 1115 = 0$ .
24.  $y = 6x - 27$ .
25.  $y = mx \pm d\sqrt{1 + m^2}$ .
26.  $Bx = A(y - b)$ .
27.  $ax - by = a^2 - b^2$ .
28.  $(a \pm b)y + (b \mp a)(x - a) = 0$ .
30.  $\begin{cases} x - 3y + 26 = 0, \\ 5x + 3y + 8 = 0, \\ 2x + 3y - 9 = 0. \end{cases}$
31.  $x - 6 = 0$ .
32.  $\begin{cases} 2x - 9y + 12 = 0, \\ 10x - 4y + 63 = 0, \\ 18x - 40y + 111 = 0. \end{cases}$
33.  $\begin{cases} x - y - 6 = 0 \\ 2x - y - 2 = 0 \\ 5x - 3y - 10 = 0 \end{cases} \begin{cases} \text{meeting in the point} \\ (-4, -10). \\ \text{Distance} = \sqrt{85}. \end{cases}$
35.  $y - y_1 = \frac{-A \pm B \tan \phi}{B \pm A \tan \phi} (x - x_1)$ .

## Ex. 15. Page 52.

1.  $\frac{1}{2}\sqrt{10}$ .      2.  $\frac{4}{3}\sqrt{5}$ .      3. 4.      4.  $\frac{4}{3}\sqrt{5}$ .      5. 0.
7.  $-\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3},$       The learner should construct the given  
 $-\frac{2}{3}, -\frac{4}{3}, 0, +\frac{4}{3},$  lines, and observe how the *sign* of the re-  
 $-\frac{1}{3}, -\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3},$  quired distance gives the direction of the  
 $-\frac{2}{3}, 0, +\frac{2}{3}.$  point from the line.
8.  $-6, -5, -4, 3, 2, 1, 0, -1.$  The learner should construct the lines, and observe the change of sign of the distance, as in No. 7.

9.  $-\frac{2}{3}\sqrt{10}$ .      17.  $\sqrt{a^2 + b^2}$ .      22. 4.
10.  $\frac{20}{11}\sqrt{41}$ .      18.  $\frac{\pm ab}{\sqrt{a^2 + b^2}}; \frac{\mp 3ab}{\sqrt{a^2 + b^2}}$ .      23.  $\pm \frac{C - C'}{\sqrt{A^2 + B^2}}$ .
11.  $2\sqrt{2}$ .      19.  $\sqrt{a^2 + b^2}$ .      24.  $\frac{C + C'}{\sqrt{A^2 + B^2}}$ .
12.  $-\frac{2}{3}\sqrt{2}$ .      20.  $\pm \frac{Ah + Bk - (D - C)}{\sqrt{A^2 + B^2}}$ .      25.  $\frac{1}{15}\sqrt{26}$ .
13.  $-\frac{11}{5}$ .      21. 2.      26.  $\pm \frac{3ab}{2\sqrt{a^2 + b^2}}$ .
14.  $\frac{6}{13}\sqrt{13}$ .      25.  $\frac{1}{15}\sqrt{26}$ .
15.  $\frac{31}{5}\sqrt{2}$ .
16.  $\frac{4}{3}\sqrt{2}$ .

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## Ex. 16. Page 54.

1.  $1\frac{1}{2}$ .      4. 40.      8. 35.      11. 26.
2. 12.      5.  $ab$ .      9.  $19\frac{1}{2}$ .      12. 96.
3. 29.      7. 26.      10.  $\frac{1}{2}(x_1y_2 - x_2y_1)$ .      13. 41.
14.  $\frac{1}{2}(a - c)(b - 1)$ .      21.  $9a^2$ .      27.  $\frac{b^2}{2m}$ .
15.  $\frac{1}{2}(a - b)(2c - a - b)$ .      22.  $\frac{2c^2}{21}$ .      28.  $\frac{1}{2}ab$ .
16.  $\frac{1}{2}(b^2 - a^2)$ .      23. 24.      29.  $\frac{C^2}{2AB}$ .
17.  $60^\circ, 60^\circ, 60^\circ; 9\sqrt{3}$ .      24. 36.      30. 56.
18. 10.      25. 16.      31.  $10\frac{1}{2}$ .
19.  $\frac{1}{4}$ .      26.  $\frac{1}{2}ab$ .
20.  $1\frac{1}{2}$ .

## Ex. 17. Page 56.

3. 2,  $\infty$ ,  $90^\circ$ , 2,  $0^\circ$ .      4. 0, 0,  $45^\circ$ , 0,  $135^\circ$ .
5.  $\frac{1}{3}\sqrt{3}-2$ ,  $2\sqrt{3}-1$ ,  $60^\circ$ ,  $\frac{2\sqrt{3}-1}{2}$ ,  $150^\circ$ .
6. 2,  $\frac{2}{3}\sqrt{3}$ ,  $150^\circ$ , 1,  $60^\circ$ .      26.  $4y = x + 8$ .
7. 2,  $-\frac{2}{3}\sqrt{3}$ ,  $30^\circ$ , 1,  $300^\circ$ .      27.  $4y = 9x - 24$ .
8.  $\frac{2}{3}\sqrt{3}$ , -2,  $60^\circ$ , 1,  $330^\circ$ .      28.  $\begin{cases} 9x - 20y + 96 = 0, \\ 5x - 4y + 32 = 0. \end{cases}$
9.  $\begin{cases} 11x + y = 0, \\ x - 5y + 20 = 0. \end{cases}$       29.  $88x - 121y + 371 = 0$ .
10.  $\frac{4}{41}\sqrt{82}$ .      30.  $\begin{cases} 5x - y - 10 = 0, \\ x + 5y - 28 = 0. \end{cases}$
11.  $\begin{cases} 3x + 4y - 57 = 0, \\ 3x + 4y + 6 = 0, \\ 12x - 5y - 39 = 0, \\ 12x - 5y + 24 = 0. \\ \text{area} = 63. \end{cases}$       31.  $\begin{cases} 2x + y - 9 = 0, \\ x - 2y - 17 = 0. \end{cases}$
12. 43.      32.  $\begin{cases} 4x + y - 20 = 0, \\ x - 4y - 5 = 0. \end{cases}$
13.  $x = 3$ .      33.  $2x = y$ ,  $2y = x$ .
14.  $\begin{cases} x - y + 1 = 0, \\ x + y - 7 = 0. \end{cases}$       34.  $4x + 5y + 11 \pm 3\sqrt{41} = 0$ .
15.  $5x + 6y - 39 = 0$ .      35.  $y = (7 \pm 5\sqrt{2})(x + 2)$ .
16.  $14x - 3y - 30 = 0$ .      36.  $\frac{2x - 5y}{\sqrt{29}} = \pm \frac{4x + 3y - 12}{5}$ .
17.  $4x - 5y + 8 = 0$ .      37.  $\begin{cases} 7x - 3y + 15 = 0, \\ 3x + 7y - 93 = 0. \end{cases}$
18.  $x + y - 7 = 0$ .      38.  $\begin{cases} 8x + 7y - 19 = 0, \\ 16x + 3y + 17 = 0. \end{cases}$
19.  $\frac{y - y_2}{x - x_2} = \frac{y_2 - y_1}{x_2 - x_1}$ .      39.  $135^\circ$ .
20.  $\begin{cases} y = 3, \quad 13y = 5x - 1, \\ 9y = 5x + 7. \end{cases}$       40.  $90^\circ$ .
21.  $92x + 69y + 102 = 0$ .      41.  $-\frac{61\sqrt{26}}{143}$ .
22.  $x + 4y = 34$ .      42.  $\pm \frac{bh + ak - ab}{\sqrt{a^2 + b^2}}$ .
23.  $3x + 4y - 5a = 0$ .      43.  $\frac{c^2}{\sqrt{h^2 + k^2}}$ .
24.  $3x + 4y = 24$ .      44.  $\pm \frac{a}{m} \sqrt{1 + m^2}$ .
25.  $y - y_1 = -\frac{y_1}{x_1}(x - x_1)$ .      45.  $c^2$ .



46.  $\frac{k^2}{6}$ .

47.  $\frac{2a^2 + 5ab + 2b^2}{6}$ .

48.  $17\frac{1}{2}$ .

49. ~~25~~ 1.

50. 59.

51.  $(10, 5\frac{1}{2})$ .

62. If  $h$  denotes the altitude of the triangle, and the base is taken as the axis of  $x$ , the locus is the straight line  $y = h$ .

63. The equation of the locus is

$$(x - x_1)^2 + (y - y_1)^2 = (x - x_2)^2 + (y - y_2)^2.$$

This is the equation of the straight line bisecting the line joining  $(x_1, y_1)$  and  $(x_2, y_2)$ , and  $\perp$  to it.

64. The two parallel lines represented by

$$Ax + By + C \pm d\sqrt{A^2 + B^2} = 0.$$

65.  $x + y = k$ .

66.  $\frac{Ax + By + C}{\sqrt{A^2 + B^2}} + \frac{A'x + B'y + C'}{\sqrt{A'^2 + B'^2}} = k$ .

67. Let  $b$  denote the base,  $k^2$  the constant difference of the squares of the other two sides. Taking the base as axis of  $x$ , and middle point of the base as origin, the equation of the locus is  $2bx = \pm k^2$ .

### Ex. 18. Page 64.

1.  $7x + y = 0$ .

2.  $x + 2y - 13 = 0$ .

3.  $5x + 6y - 37 = 0$ .

4.  $\begin{cases} x - y + 8 = 0, \\ x + y - 6 = 0. \end{cases}$

5.  $y = x + 3$ .

6.  $64x - 23y = 59$ .

7.  $44x + y = 0$ .

8.  $5x + y - 16 = 0$ .

9.  $(AC' - A'C)x + (BC' - B'C)y = 0$ .

10.  $(BA' - AB')y + CA' - AC' = 0$ .

11.  $\frac{Ax + By + C}{Ax_1 + By_1 + C} = \frac{A'x + B'y + C'}{A'x_1 + B'y_1 + C'}$ .

12.  $472x - 29y + 174 = 0$ .

13.  $y = x\sqrt{3} + 3 - \sqrt{3}$ .

14.  $\begin{cases} 4x + 3y - 25 = 0, \\ 3x - 4y + 25 = 0. \end{cases}$

15.  $\frac{y}{a} - \frac{x}{b} = \frac{mb - a}{ma + b}$ .

16-18. Generally the easiest way to solve such exercises as these is to find the intersection of two of the lines, and then substitute its co-ordinates in the equation of the third line.

19.  $m = 1$ .

20. When  $\frac{m'' - m}{m'' - m'} = \frac{b'' - b}{b'' - b'}$ .

21. If we choose as axes one side of the triangle and the corresponding altitude, we may represent the three vertices by  $(a, 0)$ ,  $(-c, 0)$ ,  $(0, b)$ .

22. Choosing as axes one side and the perpendicular erected at its middle point, the vertices may be represented by  $(a, 0)$ ,  $(-a, 0)$ ,  $(b, c)$ .

23. It is well here to choose the same axes as in No. 21.

24. Choosing the origin anywhere within the triangle, it is evident that the equations of the bisectors in the normal form may be written as follows:

$$\begin{aligned}(x \cos \alpha + y \sin \alpha - p) - (x \cos \alpha' + y \sin \alpha' - p') &= 0, \\(x \cos \alpha' + y \sin \alpha' - p') - (x \cos \alpha'' + y \sin \alpha'' - p'') &= 0, \\(x \cos \alpha'' + y \sin \alpha'' - p'') - (x \cos \alpha + y \sin \alpha - p) &= 0.\end{aligned}$$

Now, by adding any two of these equations, we obtain the third; therefore the three bisectors must pass through one point.

25.  $\begin{cases} 2\sqrt{2}, \sqrt{10}, 2\sqrt{10}. \\ \text{Origin within the } \Delta. \end{cases}$

29.  $\begin{cases} x - y + 2 = 0, \\ x + y - 14 = 0. \end{cases}$

26.  $\frac{1}{2}\sqrt{10}, \frac{1}{3}\sqrt{34}, \frac{1}{3}\sqrt{13}$ .

30.  $\begin{cases} x - 1 = 0, \\ y - 1 = 0. \end{cases}$

27.  $\begin{cases} x + y + 10 = 0, \\ 7x - 7y + 24 = 0. \end{cases}$

28.  $\begin{cases} 7x - 9y + 34 = 0, \\ 9x + 7y - 12 = 0. \end{cases}$

31.  $\frac{\pm(y - mx - b)}{\sqrt{1 + m^2}} = \pm \frac{\pm(y - m'x - b')}{\sqrt{1 + m'^2}}$ .

## Ex. 19. Page 68.

1. (i.) Parallel to the axis of  $x$ , (ii.) parallel to the axis of  $y$ .

2. When  $ad = bc$ .

3. The two lines are *real*, *imaginary*, or *coincident*, according as  $C^2 - 4AB$  is *positive*, *negative*, or *zero*. The two lines are  $\perp$  to each other when  $A + B = 0$ .

5.  $x + y + 1 = 0$ , and  $x - 3y + 1 = 0$ .

6.  $x - 2y \pm (y - 3)\sqrt{-1} = 0$ .

7.  $x - y - 3 = 0$ , and  $x - 3y + 3 = 0$ .

8.  $45^\circ$ .

9.  $K = 2$ .

10.  $K = -10$  or  $-\frac{3}{2}$ .

11.  $K = 28$ .

12.  $K = \frac{1}{4}$ .

Ex. 20. Page 70.

1. Take the point  $O$  as origin, and the axis of  $y$  parallel to the given lines. If the equations of the given lines are  $x=a$ ,  $x=b$ , and if the slopes of the lines drawn in the two fixed directions are denoted by  $m'$ ,  $m''$ , the equation of the locus is

$$(b-a)y = m'b(x-a) - m''a(x-b).$$

2. If  $a$  and  $b$  are the segments of the hypotenuse made by a perpendicular dropped from the vertex of the right angle, the equation of the locus is

3. Let  $OA=a$ ,  $OB=b$ . Then the equation of the locus is  $x+y=a+b$ .

4. Take as axes the base and the altitude of the triangle. Let  $b$  denote the base,  $a$  one segment of the base,  $h$  the altitude. Then the equation of the locus is

$$\frac{2x}{b-a} + \frac{2y}{h} = 1.$$

This is a straight line joining the middle points of the base and the altitude.

5. Take as axes the sides of the rectangle, and let  $a$ ,  $b$  denote their lengths. The equation of the locus is

$$bx - ay = 0.$$

Hence the locus is a diagonal of the rectangle.

Ex. 21. Page 73.

- |   |   |
|---|---|
| 1. $x^2 + y^2 = -2rx$ .                                     | 13. $(4, 0)$ , 4.   |
| 2. $x^2 + y^2 = 2ry$ .                                      | 14. $(-4, 0)$ , 4.  |
| 3. $x^2 + y^2 = -2ry$ .                                     | 15. $(0, 4)$ , 4.   |
| 4. $(x-5)^2 + (y+3)^2 = 100$ .                              | 16. $(0, -4)$ , 4.  |
| 5. $x^2 + (y+2)^2 = 121$ .                                  | 17. $(0, \frac{7}{2})$ , $\frac{7}{2}$ .                          |
| 6. $(x-5)^2 + y^2 = 25$ .                                   | 18. $(0, 0)$ , $3k$ .   |
| 7. $(x+5)^2 + y^2 = 25$ .                                   | 19. $(0, 0)$ , $2k$ .   |
| 8. $(x-2)^2 + (y-3)^2 = 25$ .                               | 20. $(0, 0)$ , $\sqrt{a^2 + b^2}$ .                               |
| 9. $x^2 + y^2 - 2hx - 2ky = 0$ .                            | 21. $(\frac{k}{2}, 0)$ , $\frac{k}{2}\sqrt{5}$ .                  |
| 11. $(1, 2)$ , $\sqrt{5}$ .                                 | 22. $(\frac{h}{2}, \frac{k}{2})$ , $\frac{\sqrt{h^2 + k^2}}{2}$ . |
| 12. $(\frac{1}{2}, \frac{1}{2})$ , $\frac{1}{2}\sqrt{62}$ . |   |

$a$  and  $b$  denote the segments

23. When  $D = D'$  and  $E = E'$ ; in other words, when the two equations differ only in their constant terms.

24. In this case  $r = 0$ . Hence the equation represents simply the point  $(a, b)$ . We may also say that it is the equation of an infinitely small circle, having this point for centre.

$$26. \begin{cases} (\frac{1}{2}, \frac{1}{2}), \frac{1}{2}\sqrt{2}; \\ \text{On } OX, 3 \text{ and } 2; \\ \text{On } OY, 6 \text{ and } 1. \end{cases}$$

$$31. \begin{cases} \text{(i.) } D^2 = 4C. \\ \text{(ii.) } E^2 = 4C. \\ \text{(iii.) } 4C > D^2 \text{ and } E^2. \end{cases}$$

$$27. \begin{cases} (6, 2), 5; \\ \text{On } OX, 6 \pm \sqrt{21}; \\ \text{On } OY, \text{imaginary points.} \end{cases}$$

$$32. x^2 + y^2 + 10x + 10y + 25 = 0.$$

$$33. (7, 4) \text{ and } (8, 1).$$

$$28. \begin{cases} (2, 4), 2\sqrt{5}; \\ \text{On } OX, 0 \text{ and } 4; \\ \text{On } OY, 0 \text{ and } 8. \end{cases}$$

$$34. (2, 0) \text{ and } (\frac{8}{3}, -\frac{8}{3}).$$

$$35. \frac{4}{3}\sqrt{5}.$$

$$29. \begin{cases} (3, -2), 3; \\ \text{On } OX, 3 \pm \sqrt{5}; \\ \text{On } OY, -2. \end{cases}$$

$$36. 2\left(r^2 - \frac{a^2b^2}{a^2 + b^2}\right)^{\frac{1}{2}}.$$

$$37. 2x - y - 2 = 0.$$

$$30. \begin{cases} (-11, 9), \sqrt{145}; \\ \text{On } OX, -3 \text{ and } -19; \\ \text{On } OY, 9 \pm 2\sqrt{6}. \end{cases}$$

$$38. 4x - 5y - 71 = 0.$$

$$39. 3x - 5y - 34 = 0.$$

40. Let  $(x, y)$  be any point in the required locus; then the distance of  $(x, y)$  from  $(x_1, y_1)$  must always be equal to its distance from  $(x_2, y_2)$ ;

therefore  $(x - x_1)^2 + (y - y_1)^2 = (x - x_2)^2 + (y - y_2)^2$ ;

whence  $2x(x_1 - x_2) + 2y(y_1 - y_2) = (x_1^2 + y_1^2 - x_2^2 - y_2^2)$ .

Show that this represents a straight line  $\perp$  to the line joining  $(x_1, y_1)$  and  $(x_2, y_2)$ .

$$41. 8x + 6y + 17 = 0.$$

42. FIRST METHOD. Substitute successively the co-ordinates of the given points in the general equation of the circle; this gives three equations of condition, and by solving them we find the values of  $a, b, r$ .

SECOND METHOD. Join  $(4, 0)$  to  $(0, 4)$  and also to  $(6, 4)$  by straight lines, then erect perpendiculars at the middle points of these two lines; their intersection will be the centre of the circle, and the distance from the centre to either one of the given points will be the radius.

$$\text{Ans. } x^2 + y^2 - 6x - 6y + 8 = 0.$$

$$43. x^2 + y^2 - 8x + 6y = 0.$$

$$45. x^2 + y^2 + 8ax - 6ay = 0.$$

$$44. x^2 + y^2 + 6x + y = 0.$$

$$46. x^2 + y^2 + 8x + 20y + 31 = 0.$$

47.  $x^2 + y^2 - 9x - 5y + 14 = 0$ . 51.  $x^2 + y^2 + 2ax + 2ay + a^2 = 0$ .  
 48.  $\begin{cases} (x-5)^2 + (y+8)^2 = 169, \\ (x-22)^2 + (y-9)^2 = 169. \end{cases}$  52.  $x^2 + y^2 = ax + by$ .  
 49.  $\begin{cases} x^2 + y^2 - 30(x+y) + 225 = 0. \\ x^2 + y^2 - 6(x+y) + 9 = 0. \end{cases}$  53.  $(x-1)^2 + (y-4)^2 = 20$ .  
 50.  $x^2 + y^2 - 8x - 8y + 16 = 0$ . 54.  $x^2 + y^2 - 14x - 4y - 5 = 0$ .  
 55.  $x^2 + y^2 \pm \sqrt{2}ay = 0$ ; also  $x^2 + y^2 \pm \sqrt{2}ax = 0$ .  
 56.  $m(x^2 + y^2) - ab = (ma - b)x + (mb - a)y$ . 57.  $x^2 + y^2 = x_1x + y_1y$ .  
 58.  $(x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0$ . 60.  $x^2 - ax + y^2 = r^2 - \frac{a^2}{2}$ .  
 59.  $(1+m^2)(x^2 + y^2) - 2r(x+my) = 0$ .

## Ex. 22. Page 81.

1. The double sign corresponds to the geometric fact that *two* tangents having the same direction may always be drawn to a given circle.  
 3.  $2x + 3y = 26$ ,  $3x - 2y = 0$ ;  $\sqrt{117}$ ,  $2\sqrt{13}$ , 9, 4,  $\frac{1}{13}\sqrt{13}$ .  
 4.  $\frac{x_1^2 - r^2}{x_1}$ ,  $-x_1$ ,  $\frac{r^2}{x_1 y_1}$ . 22.  $\begin{cases} x^2 + y^2 = p^2, \\ (p \cos \alpha, p \sin \alpha). \end{cases}$   
 5.  $9x - 13y = 250$ . 23.  $\begin{cases} \text{When } C = r(A^2 + B^2)^{\frac{1}{2}}; \\ \text{When } \frac{Aa + Bb}{\sqrt{A^2 + B^2}} = \pm r. \end{cases}$   
 6.  $x \pm 3y = 10$ . 24.  $ax + by = 0$ .  
 7.  $104\frac{1}{2}$ . 25.  $(-a, -b)$ .  
 8.  $x^2 + y^2 = 25\frac{1}{2}$ . 26.  $(2a, b)$ .  
 9.  $14x \pm 6y = 232$ . 27.  $(0, b)$ .  
 10.  $3x + y = 19$ . 28.  $x^2 + y^2 = \frac{5}{2}$ .  
 11.  $3x + 4y = 0$ . 29.  $m = 0$ .  
 12.  $\begin{cases} 3x + 7y = 93, \\ 3x - 7y = 65. \end{cases}$  30.  $c = -36 \pm 20\sqrt{6}$ .  
 13.  $x = r$ . 31.  $(x-5)^2 + (y-3)^2 = \frac{1}{13}$ .  
 14.  $Ax + By \pm r\sqrt{A^2 + B^2} = 0$ . 32.  $\begin{cases} (x-2)^2 + (y-4)^2 = 100, \\ (x-18)^2 + (y-16)^2 = 100. \end{cases}$   
 15.  $Bx - Ay \pm r\sqrt{A^2 + B^2} = 0$ . 33.  $(x-1)^2 + (y-6)^2 = 25$ .  
 16.  $x - y \pm r\sqrt{2} = 0$ . 34.  $\frac{1}{r^2} = \frac{1}{a^2} + \frac{1}{b^2}$ .  
 17. The equation of the two tangents is  $(h^2 - r^2)y^2 = r^2(x-h)^2$ . 35.  $(x^2 + y^2)(a + b + \sqrt{a^2 + b^2})^2 - 2ab(a + b + \sqrt{a^2 + b^2})(x + y) + a^2b^2 = 0$ .  
 18.  $x + y = r\sqrt{2}$ . 36.  $x = a + r$ .  
 19.  $\begin{cases} x = 10, \\ 3x + 4y = 50. \end{cases}$  37.  $[4r^2 - 2(a-b)^2]^{\frac{1}{2}}$ .  
 20.  $y = 2x + 13 \pm 6\sqrt{5}$ .  
 21.  $-21, -3\frac{1}{2}$ .

## Ex. 23. Page 84.

1.  $(1, -\frac{2}{3}), \frac{1}{2}\sqrt{29}$ .
2.  $(-\frac{2}{3}, \frac{2}{3}), \frac{1}{2}\sqrt{11}$ .
3.  $(\frac{2}{3}, \frac{2}{3}), \sqrt{\frac{17}{2}}$ .
4.  $(\frac{b}{\sqrt{1+a^2}}, \frac{ab}{\sqrt{1+a^2}}), b$ .
5.  $x^2 + y^2 = 81$ .
6.  $(x-7)^2 + y^2 = 9$ .
7.  $(x+2)^2 + (y-5)^2 = 100$ .
8.  $x^2 + y^2 - 2a(3x+4y) = 0$ .
9.  $x^2 + y^2 + 2b^2 + c^2 = 2[(b+c)x + (b-c)y]$ .
10.  $3ab(x^2+y^2) + 2ab(a^2+b^2) = (5a^2+2b^2)bx + (5b^2+2a^2)ay$ .
11.  $x^2 + y^2 - 5x - 12y = 0$ .
12.  $x^2 + y^2 - 14x - 4y - 5 = 0$ .
13.  $x^2 + y^2 + 14x + 14y + 49 = 0$ .
14.  $x^2 + y^2 \mp 2rx \mp 2ry + r^2 = 0$ ,  
 $x^2 + y^2 \pm 2rx \mp 2ry + r^2 = 0$ .
15.  $x^2 + y^2 - 2ax - 2ay + a^2 - \frac{b^2}{4} = 0$ .
16.  $x^2 + y^2 = \frac{2}{5}$ .
17.  $5(x^2+y^2) - 10x + 30y + 49 = 0$ .
18.  $\begin{cases} (x-3)^2 + (y-1)^2 = 5, \\ (x+\frac{17}{6})^2 + (y+\frac{13}{6})^2 = \frac{5}{4}. \end{cases}$
19.  $x^2 + y^2 - 30 - 52y = 0$ .
20.  $x^2 + y^2 + 50x + 88y - 50 = 0$ .
21.  $\begin{cases} x^2 + y^2 - 36x - 46y + 324 = 0, \\ 25x^2 + 25y^2 - 80x - 494y + 64 = 0. \end{cases}$
22.  $(6, 2), 5$ .
23.  $-15$ .
24.  $-10$ .
25.  $\frac{1}{2}\sqrt{234}$ .
26.  $\sqrt{10}$ .
27.  $x_1x + y_1y = x_1^2 + y_1^2$ .
28. (i.)  $D^2 = 4AC$ , (ii.)  $E^2 = 4AC$ ,  
(iii.)  $D^2 = E^2 = 4AC$ .
29.  $r^2 = 2rmc + c^2$ .
30.  $k = 40$ , or  $-10$ .
31.  $x^2 + y^2 = ay\sqrt{2}$ .
32.  $x^2 + y^2 \pm 2a(x \pm y) = 0$ .
33.  $2(x^2 - ax + y^2 - r^2) + a^2 = 0$ .
34.  $x - y = 0$ .
35.  $4x + 3y = 0$ .
36.  $(18 \pm 2\sqrt{41})x - 5y = 0$ .
37.  $x + \sqrt{3}y \pm 20 = 0$ .
38.  $x + y - 10 = 0$ .
40.  $\frac{1}{2}(35 + 24\sqrt{30})$ .
43.  $135^\circ$ .
44.  $(7, -5)$  and  $(-6\frac{7}{8}, 9\frac{1}{8})$ .
45.  $\begin{cases} (x+4)^2 + (y+10)^2 = 85, \\ (x - \frac{514}{169})^2 + (y + \frac{670}{169})^2 = \frac{85}{169^2}. \end{cases}$
46. The circle  $(x-x_1)^2 + (y-y_1)^2 = r^2$ .
47. The circle  $(x-a)^2 + (y-b)^2 = (r+r')^2$ ,  
 $(x-a)^2 + (y-b)^2 = (r-r')^2$ .
48. The circle  $(x-a)^2 + (y-b)^2 = r^2 + t^2$ .

49. Take  $A$  as origin, and let the radius of the circle  $= r$ ; then the locus is the circle  $x^2 + y^2 = rx$ .

50. Take  $A$  as origin, and let the radius of the circle  $= r$ ; then the locus is the circle  $x^2 + y^2 = \frac{2mrx}{m+n}$ .

51. Take  $A$  as origin,  $AB$  as axis of  $x$ , and let  $AB = a$ ; then the locus is the circle  $(m^2 - n^2)(x^2 + y^2) - 2am^2x + a^2m^2 = 0$ .

52. Take  $AB$  as the axis of  $x$ , the middle point of  $AB$  as origin, and let  $AB = 2a$ ; then the locus is the circle  $2(x^2 + y^2) = k^2 - 2a^2$ .

53. Using the same notation as in No. 52, the locus is the straight line  $4ax = k^2$ .

54. Taking the fixed lines as axes, the locus is the circle  $4(x^2 + y^2) = d^2$ .

55. Take the base as axis of  $x$ , its middle point as origin, and let the length of the base  $= 2a$ , and the constant angle at the vertex  $= \theta$ . Then the locus is the circle  $x^2 + y^2 - 2a \cot \theta y = a^2$ .

56. Take  $A$  as origin,  $AB$  as axis of  $x$ , and let  $AB = a$ ,  $AC = b$ . Then the locus is the circle  $(x - \frac{1}{2}a)^2 + y^2 = \frac{b^2}{4}$ .

57. The circle  $x^2 + y^2 = \frac{4r^2}{4r^2 - l^2}$ , where  $l$  is the length of the chord.

58. The locus consists of the two circles  $x^2 + y^2 \pm rx = 0$ .

#### Ex. 24. Page 97.

1.  $7x - 6y = 0$ .
2.  $x - y = 0$ .
4.  $x + y = r$ ,  $2x + 3y = r$ ,  $(a + b)x + (a - b)y = r^2$ .
5.  $13x + 2y = 49$ .
6. The tangent at  $(h, k)$ .
7. (i.)  $2x + 3y = 4$ , (ii.)  $3x - y = 4$ , (iii.)  $x - y = 4$ .
8. (i.)  $(20, 30)$ , (ii.)  $(21, -14)$ , (iii.)  $(35a, 35b)$ .
9.  $(6, 8)$ .
18.  $12x + 17y - 51 = 0$ .
10.  $\left(-\frac{Ar^2}{C}, -\frac{Br^2}{C}\right)$ .
19.  $x + y - 2 = 0$ .
11.  $(4, \pm 3)$ ,  $4x \pm 3y = 25$ .
20.  $(a^2 - ab)x - (ab - b^2)y + ac = 0$ .
16.  $h^2 + k^2 - r^2$ .
21.  $x - y = 0$ ,  $\sqrt{\frac{1}{2}(a + b)^2 - 4c}$ .
17. 3.
22.  $(-2, -1)$ .

## Ex. 26. Page 109.

1. Writing  $x + 1$  for  $x$ , and  $y - 2$  for  $y$ , and reducing, we have  $y^2 = 4x$ .
2.  $x^2 + y^2 = r^2$ .
3.  $x^2 + y^2 = 2rx$ .
4.  $x^2 + y^2 = -2ry$ .
5.  $x^2 + y^2 = r^2$ .
6.  $2xy = a^2$ .
7.  $x^2 - y^2 + 2 = 0$ .
8. (i.)  $\rho = \pm a$ , (ii.)  $\rho^2 \cos 2\theta = a^2$ .
9. (i.)  $\rho = 4a \tan \theta \sec \theta$ , (ii.)  $(a + \rho \cos \theta)^2 = 4a\rho \sin \theta$ .
10. (i.)  $x^2 + y^2 = a^2$ , (ii.)  $x^2 + y^2 = ax$ , (iii.)  $x^2 - y^2 = a^2$ .
11.  $x + y = 0$ .
12.  $2x - 5y + 10 = 0$ .
13.  $12x^2 + 16xy + 4y^2 + 1 = 0$ .
14.  $x^2 + y^2 = 25$ .
15.  $x^2 - 6xy + y^2 = 0$ .
16.  $xy = 3$ .
17.  $y^2 = 2a(x\sqrt{2} - a)$ .
18.  $4xy = 25$ .

## Ex. 27. Page 111.

1.  $b\sqrt{3}$ .
2.  $4 \sin \frac{1}{2} \omega$ .
3.  $\sqrt{13 - 12 \cos \omega}$ .
4.  $\sqrt{a^2 + b^2 - 2ab \cos(\theta - \phi)}$ .
5.  $2a \sin \theta$ .
6.  $2a \cos \theta$ .
7.  $a\sqrt{5 - 2\sqrt{3}}$ .
9.  $2x^2 + 2xy + y^2 = 1$ .
10.  $2x^2 + y^2 = 6$ .
11.  $y = 0$ .
12.  $9x^2 + 25y^2 = 225$ .
13.  $\rho = 8a \cos \theta$ .
14.  $\rho = \pm 4a$ .
15.  $\rho^2 \sin^2 \theta - 5\rho \cos \theta = \frac{25}{4}$ .
16.  $\rho^2 = 49 \sec 2\theta$ .
17.  $\rho^2 = k^2 \cos 2\theta$ .
18.  $xy = a^2$ .
19.  $(x^2 + y^2)^{\frac{3}{2}} = 2kxy$ .
20.  $x^3 - y^3 + (3x - 3y - 5k)xy = 0$ .
21.  $\tan^{-1} \frac{7}{5}$ .
22. (i.)  $\tan^{-1} \left( -\frac{A}{B} \right)$ , (ii.)  $\tan^{-1} \frac{B}{A}$ .

## Ex. 28. Page 117.

2.  $y^2 = 4px - 4p^2$ .
3.  $y^2 = 4px + 4p^2$ .
4. (i.)  $y^2 = 10x$ , (ii.)  $y^2 = 10x + 25$ , (iii.)  $y^2 = 10x - 25$ .
5. (i.)  $y^2 = 16x$ , (ii.)  $y^2 = 16x + 64$ , (iii.)  $y^2 = 16x - 64$ .



6. (2, 6).

8. (4, 6) and (25, 15).

7. 6, 15,  $\frac{a}{b}$ .

9. (12, 6).

10. The line  $x = 9$  meets the parabola in (9, 6) and (9, -6). The line  $x = 0$  passes through the vertex. The line  $x = -2$  does not meet the parabola.

11. The line  $y = 6$  meets the parabola in (9, 6). The line  $y = -8$  meets the parabola in (16, -8).

12.  $p = 4$ .

13. The point (2, 8).

14. (i.)  $y = 0$ , (ii.)  $x = -2$ , (iii.)  $x = 2$ , (iv.)  $4x + 3y - 8 = 0$ , (v.)  $y = -2x$ .

15. (i.)  $4x - 5y + 24 = 0$ , (ii.)  $x^2 + y^2 - 20x = 0$ .

16.  $3p$ .

17.  $8p\sqrt{3}$ .

24. The latus rectum of each  $= 4p$ . The common vertex is at the origin. The axis of  $x$  is the axis of (i.) and (ii.); that of  $y$  is the axis of (iii.) and (iv.). Parabola (i.) lies wholly to the *right* of the origin, (ii.) wholly to the *left*, (iii.) wholly *above*, (iv.) wholly *below*. We may name them as follows:—

(i.) is a right-handed  $X$ -parabola.

(iii.) is an upward  $Y$ -parabola.

(ii.) is a left-handed  $X$ -parabola.

(iv.) is a downward  $Y$ -parabola.

### Ex. 29. Page 121.

6.  $x - 4y + 20 = 0$ ,  $4x + y - 90 = 0$ .

7. Tangents  $\begin{cases} x - y + 3 = 0, \\ x + y + 3 = 0; \end{cases}$  normals  $\begin{cases} x + y - 9 = 0, \\ x - y - 9 = 0. \end{cases}$

These lines enclose a square whose area  $= 72$ .

8. Tangent  $= \sqrt{266}$ , normal  $= \sqrt{95}$ , subtangent  $= 14$ , subnormal  $= 5$ .

9. (5, 10).

13.  $\frac{p}{m\sqrt{1+m^2}}$ ,  $\frac{-p}{m}\sqrt{m^2+1}$ .

14.  $[\sqrt{x_1x_2}, \frac{1}{2}(y_1+y_2)]$ .

15.  $x + y + p = 0$ , point of contact  $(p, -2p)$ , intercept  $= -p$ .

16. Equations of the tangents  $y\sqrt{3} = \pm x \pm 3p$ , required point  $(-3p, 0)$ .

17. For the two points whose co-ordinates are

$$x = \frac{p}{8}(1 \pm \sqrt{17}), y = \pm p\sqrt{\frac{1 \pm \sqrt{17}}{2}}.$$

18. For the points (0, 0) and  $(3p, \pm 2p\sqrt{3})$ .

19.  $9x - 6y + 5 = 0$ ,  $(\frac{5}{9}, \frac{5}{3})$ .

20.  $x - 2y + 12 = 0$ , (12, 12).

$$4x + 2y + 3 = 0, (\frac{3}{4}, -\frac{3}{2}).$$

$$21. y = x(\pm\sqrt{2} - 1) + 4(\pm\sqrt{2} + 1).$$

$$22. \frac{4\sqrt{p(p+x_1)^3}}{x_1}.$$

24. By the secant method we find that the equation of the tangent at  $(x_1, y_1)$  is

$$\frac{y - y_1}{x - x_1} = \frac{4}{y_1 - 3}.$$

The points of contact are  $(-1, 11)$  and  $(-1, -5)$ ; hence the tangents are  
 $x - 2y + 23 = 0$   
 and  $x + 2y + 11 = 0.$

$$25. \begin{cases} \text{(i.) } y_1 y = -2p(x + x_1), \\ \text{(ii.) } x_1 x = 2p(y + y_1), \\ \text{(iii.) } x_1 x = -2p(y + y_1). \end{cases}$$

### Ex. 30. Page 123.

$$1. y^2 = 24x - 144.$$

$$2. y^2 = 16x.$$

$$3. y^2 = -17x.$$

$$4. \begin{cases} 2y^2 - 11x + 12y + 73 = 0; \text{ or,} \\ 2y^2 + 11x + 12y - 37 = 0. \end{cases}$$

$$5. (y + 7)^2 = 4(x - 3).$$

$$6. 3y^2 = 4x.$$

$$7. 2x^2 = 9y.$$

$$8. \frac{3}{2}, 8x + 3 = 0, 8x \pm 15y - 3 = 0.$$

$$10. \begin{cases} 4 \text{ on } OX; \\ 8 \text{ and } -2 \text{ on } OY. \end{cases}$$

$$20. y^2 = -9x.$$

$$21. y^2 = 8x.$$

$$11. 4(2 \pm \sqrt{3})p.$$

$$22. y^2 = \frac{4r^2 - t^2}{r}x.$$

$$12. \begin{cases} \text{(i.) } y = x + 2, \\ \text{(ii.) } 2\sqrt{2}, \\ \text{(iii.) } x + y - 6 = 0. \end{cases}$$

$$23. y^2 = \frac{n^2}{r}x.$$

$$13. x + y - 6 = 0.$$

$$24. y^2 = \frac{2n^2}{\sqrt{n^2 + t^2}}x.$$

$$14. y - y_1 = \frac{2p}{y_1}(x - x_1).$$

$$25. y^2 = 2(2r - s)x.$$

$$15. (8, 4), (2, 10).$$

$$26. 4p \sqrt{2}.$$

$$16. (2, 4), (11, 10).$$

$$27. \text{The equation of the circle is } (x - 3)^2 + (y - \frac{5}{2})^2 = \frac{25}{4}.$$

$$17. y = \frac{b \pm \sqrt{b^2 + 4ap}}{2a}x.$$

$$28. (-3p, 0).$$

$$18. \begin{cases} \text{A left-handed } X\text{-parabola.} \\ \text{Latus rectum} = 2. \\ \text{Vertex, } (-2, 0). \\ \text{Focus, } (-\frac{5}{2}, 0). \\ \text{Directrix, } x = -\frac{3}{2}. \end{cases}$$

$$29. \left(\frac{p}{3}, \pm \frac{2p}{\sqrt{3}}\right).$$

$$30. \begin{cases} (p, \pm 2p); \\ 45^\circ \text{ and } 135^\circ. \end{cases}$$

$$31. 4p^2.$$

$$19. 2a\sqrt{2}.$$

$$34. \text{The parabola } y^2 = px.$$

The loci in exercises 35-38 are parabolas, the latus rectum in each being half that of the given parabola. If the given parabola is  $y^2 = 4px$ , the equations of the loci are.

35.  $y^2 = 2px - p^2$ .

36.  $y^2 = 2px - 2p^2$ .

37.  $y^2 = 2px$ .

38.  $y^2 = 2px + 2p^2$ .

39. The straight line  $y = pk$ .

40. The parabola  $y^2 - 4px = p^2k^2$ .

41. The straight line  $kx = p$ .

42. The circle  $(x-p)^2 + y^2 = \frac{p^2}{k^2}$ .

43. Take the given line as the axis of  $y$ , and a perpendicular through the given point as the axis of  $x$ , and let the distance from the point to the line  $= a$ . The locus is the parabola  $y^2 = 2a \left( x - \frac{a}{2} \right)$ .

### Ex. 31. Page 134.

10.  $3x - 5y - 6 = 0$ .

11.  $8y - 25 = 0$ .

12.  $13x + 22y + k = 0$ .

13.  $x - y - 1 = 0$ .

$$14. \begin{cases} ay = 2px. \\ \text{The cord is parallel to the} \\ \text{tangent at the end of the} \\ \text{diameter.} \end{cases}$$

15.  $y^2 = 52x$ .

18. Writing the equation in the form  $(y-3)^2 = 8(x-2)$ , and passing to parallel axes through  $(2, 3)$  we have  $y^2 = 8x$ . 8,  $(2, 3)$ ,  $(5, 3)$ ,  $y = 3$ ,  $x = 0$ .

19.  $A$  numerically,  $\left( \frac{B^2 - 4C}{4A}, -\frac{B}{2} \right)$ ,  $y = -\frac{B}{2}$ .

20.  $B$  numerically,  $\left( -\frac{A}{2}, \frac{A^2 - 4C}{4B} \right)$ ,  $x = -\frac{A}{2}$ .

21. Take the given line as the axis of  $y$ , and a perpendicular through the centre of the given circle as the axis of  $x$ . Let the radius of the circle  $= r$ ; distance from the centre to the given line  $= a$ . There are two cases to consider, since the circles may touch the given circle either externally or internally. The two loci are the parabolas.

$$y^2 = 2(a+r)x + r^2 - a^2,$$

$$y^2 = 2(a-r)x + r^2 - a^2.$$

22. Let  $2a$  be the given base,  $ab$  the given area; take the base as axis of  $x$ , its middle point as origin; then the locus is the parabola

$$x^2 + by = a^2.$$

### Ex. 32. Page 144.

1. 5, 4, 3,  $\frac{3}{2}$ .

2.  $\sqrt{2}$ , 1, 1,  $\sqrt{\frac{1}{2}}$ .

3. 2,  $\sqrt{3}$ , 1,  $\frac{1}{2}$ .

$$4. \frac{1}{\sqrt{A}}, \frac{1}{\sqrt{B}}, \sqrt{\frac{B-A}{AB}}, \sqrt{\frac{B-A}{B}}, \text{ when } A < B.$$

5.  $\frac{4}{3}\sqrt{6}$ .  
 6.  $c = \frac{1}{2}\sqrt{3}$ .  
 7.  $4x^2 + 9y^2 = 144$ .  
 8.  $25x^2 + 169y^2 = 4225$ .  
 9.  $144x^2 + 225y^2 = 32,400$ .  
 10.  $16x^2 + 25y^2 = 1600$ .  
 11.  $25x^2 + 169y^2 = 4225$ .  
 12.  $3x^2 + 7y^2 = 115$ .  
 13.  $x^2 + 2y^2 = 100$ .  
 14.  $8x^2 + 9y^2 = 8a^2$ .  
 15.  $2 : \sqrt{3}$ .  
 16.  $x = y = \pm \frac{ab}{\sqrt{a^2 + b^2}}$ .  
 17.  $(\frac{1}{3}, \frac{4}{3}), (-\frac{5}{3}, -\frac{2}{3})$ .  
 18.  $(1, 2), (1, -2)$ .  
 19.  $(3, 1), (3, -1), (-3, 1), (-3, -1)$ .  
 20. See No. 11. The equation of the locus is  $x^2 + 4y^2 = r^2$ .  
 22. Taking as axes the two fixed lines, and putting  $AP = a$ ,  $BP = b$ , the acute angle between  $AB$  and the axis of  $x = \phi$ , we find that

$$x = a \cos \phi, \quad y = b \sin \phi.$$

Therefore  $P$  describes an ellipse whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

23. The two straight lines  $y = \pm x\sqrt{-\frac{A}{B}}$ . The locus is imaginary when the values of  $y$  are imaginary; that is, when  $A$  and  $B$  have like signs.

29. The equations of the sides are

$$x = \pm \frac{ab}{\sqrt{a^2 + b^2}}, \quad y = \pm \frac{ab}{\sqrt{a^2 + b^2}};$$

$$\text{area} = \frac{4a^2b^2}{a^2 + b^2}.$$

### Ex. 83. Page 151.

1.  $\begin{cases} 4x \pm 9y = 35, \\ 9x \mp 4y = 15. \end{cases}$   
 2.  $\begin{cases} 2x \pm 3y\sqrt{3} + 12 = 0, \\ 6x\sqrt{3} \pm 4y + 13\sqrt{3} = 0. \end{cases}$   
 3.  $\begin{cases} x + 4y = 10, \\ 4x - y - 6 = 0; \\ -8, -\frac{1}{2}. \end{cases}$   
 5.  $\frac{a^2}{m^2} + \frac{b^2}{n^2} = 1$ .  
 6.  $9x^2 + 25y^2 = 225$ .  
 7.  $2y = x \pm 10$ .  
 8.  $4x - 3y \pm \sqrt{107} = 0$ .  
 9.  $x = \pm \frac{a^2}{\sqrt{a^2 + b^2}}, \quad y = \pm \frac{b^2}{\sqrt{a^2 + b^2}}$ .  
 10. Same answers as No. 9.  
 11. ~~1. a.~~  $b^2 : a^2$ .  
 12.  $x = \pm \frac{a}{\sqrt{2}}, \quad y = \pm \frac{b}{\sqrt{2}}$ .  
 13.  $y = 4, \quad 3x + 2y = 17$ .

a

14. The equation  $\pm \sqrt{5}x \pm 3y = 0$  represents the four tangents.  
 15.  $a\sqrt{1 - e^2 \cos^2 \phi}$ .  
 16.  $\frac{1}{2}(a^2 \csc \phi \sec \phi - c^2 \cot \phi)$ .  
 17. The extremities of the latera recta.  
 19. The method of solving this question is similar to that employed in § 136. The required locus is the auxiliary circle  $x^2 + y^2 = a^2$ .

## Ex. 34. Page 152.

1.  $\begin{cases} x = 8, 40y = 9x + 72; \\ \frac{1}{9}, \frac{8}{9}. \end{cases}$
  2. Within.
  3.  $\frac{1}{\sqrt{3}}$ .
  4.  $\frac{1}{\sqrt{2}}$ .
  5.  $\frac{2\sqrt{3}}{1 + \sqrt{13}}$ .
  6.  $x + y = \pm \sqrt{a^2 + b^2}$ .
  7.  $bx + cy \pm b\sqrt{a^2 + c^2} = 0$ .
  8.  $bx + ay \pm ab\sqrt{2} = 0$ .
  9.  $\frac{x}{a} \cos \phi + \frac{y}{b} \sin \phi = 1$ .
  10.  $\frac{ab}{\sqrt{a^2 - e^2 x_1^2}}$ .
  11.  $a\sqrt{1 - e^2 \cos^2 \phi}$ .
  12.  $c^2 : b^2$ .
  13.  $a^2 - e^4 x_1^2$ .
  14.  $\sqrt{(1 - e^2)(a^2 - e^2 x_1^2)}$ .
  15.  $\tan \phi = \frac{\sqrt{1 - e^2}}{e} \frac{b}{c}$ .
  18. The locus is the minor axis produced.
  19. The ellipse  $4\left(x - \frac{a}{2}\right)^2 + y^2 = r^2$ ; centre is  $\left(\frac{a}{2}, 0\right)$ ; semi-axes are  $\frac{r}{2}$  and  $r$ .
  20. The ellipse  $a^2\left(y - \frac{b}{2}\right)^2 + b^2x^2 = \frac{a^2b^2}{4}$ ; centre is  $\left(0, \frac{b}{2}\right)$ ; semi-axes are  $\frac{a}{2}$  and  $\frac{b}{2}$ .
- In 21-23 take the base of the triangle as the axis of  $x$ , and the origin at its middle point.

21. The ellipse  $(s^2 - c^2)x^2 + s^2y^2 = s^2(s^2 - c^2)$ .
22. The ellipse  $kx^2 + y^2 = kc^2$ .
23. The circle  $(x + c)^2 + y^2 = 4a^2$ .

## Ex. 35. Page 164.

1.  $8\pi$ .
2.  $\frac{8}{3}\sqrt{3}$ .
3.  $20x + 63y - 36 = 0$ .
5.  $\left(-\frac{Aa^2}{C}, -\frac{Bb^2}{C}\right)$ .
8. (i.)  $m_1^2 = \frac{b^2}{a^2}$ , (ii.)  $m_1^2 = \frac{b}{a}$ , (iii.)  $m_1^2 = 1$ .

$$9. a = l \sqrt{\frac{1 - e^2 \cos^2 \theta}{1 - e^2}}, \quad b = l \sqrt{1 - e^2 \cos^2 \theta}.$$

$$11. 3x + 8y = 4, \quad 2x - 3y = 0.$$

12. Area =  $\frac{b^2}{2a}(m + n)$ ,  $m$  and  $n$  being the two segments (use the polar equation).

$$13. 26x + 33y - 92 = 0.$$

$$14. x + 2y = 8.$$

$$15. b^2x + a^2y = 0, \quad b^2x - a^2y = 0, \quad a^2y + b^2x = 0, \quad bx + ay = 0.$$

$$17. a^2y_1x = b^2x_1y.$$

$$29. \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{2x}{a}.$$

$$23. \frac{x}{a} \pm \frac{y}{b} = 0.$$

$$24. bx\sqrt{b^2 - b^2} \pm ay\sqrt{a^2 - b^2} = 0.$$

$$30. \rho = \frac{a(1 - e^2)}{1 + e \cos \theta}.$$

$$25. e = \frac{1}{3}\sqrt{6}.$$

$$31. \rho^2 = \frac{b^2}{1 - e^2 \cos^2 \theta}.$$

$$26. \text{See } \S 148.$$

$$32. 16x^2 + 49y^2 - 128x - 686y + 1873 = 0.$$

$$33. 2a = 18, \quad 2b = 10.$$

$$34. \frac{25x^2}{144} + y^2 = 5x.$$

$$35. \cos \phi = \sqrt{\frac{c^2 - b^2}{a^2 - b^2}}.$$

$$36. \tan \theta = \frac{a - b}{\pm \sqrt{ab}}.$$

37. Find the ratio of  $y_1$  to the intercept on the axis of  $y$ .

$$38. b^2hx + a^2ky = b^2h^2 + a^2k^2.$$

$$42. \text{The ellipse } \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{2}.$$

$$41. \text{The ellipse } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 2.$$

$$43. \text{The ellipse } b^2x^2 + a^2y^2 = b^2c^2.$$

$$44. a = 2, \quad b = 1, \quad (-1, 1).$$

$$45. \sqrt{\frac{K}{A}}, \quad \sqrt{\frac{K}{B}}, \quad \text{in which } K = -F + \frac{D^2}{4A} + \frac{E^2}{4B}.$$

$$46. \text{The ellipse } 25x^2 + 16y^2 - 48y = 64.$$

Ex. 36. Page 174.

$$1. \frac{x^2}{64} - \frac{y^2}{49} = 1.$$

$$3. 3x^2 - y^2 = 3a^2.$$

$$2. \frac{4x^2}{25} - \frac{y^2}{36} = 1.$$

$$4. 625x^2 - 84y^2 = 10,000.$$

$$5. 2x^2 - 2y^2 = c^2.$$

$$7. a = 4, \quad b = 3, \quad c = 5, \quad e = \frac{5}{4}, \quad \text{latus rectum} = \frac{3}{2}.$$



$$19. \begin{cases} \text{(i.) } \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{2x}{a} = 0. \\ \text{(ii.) } \frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{2x}{a} = 0. \end{cases} \quad \begin{array}{l} 20. \rho = \frac{a(e^2 - 1)}{1 - e \cos \theta}. \\ 21. \rho^2 = \frac{b^2}{e^2 \cos^2 \theta - 1}. \end{array}$$

23. The hyperbola  $3x^2 - y^2 + 20x - 100 = 0$ . The centre is the point  $(-\frac{10}{3}, 0)$ . Changing the origin to the centre, we obtain  $9x^2 - 3y^2 = 400$ .

24. Writing the equation in the form  $(x-1)^2 - 4(y+2)^2 = 4$ , and changing the origin to  $(1, -2)$  we obtain  $\frac{x^2}{4} - \frac{y^2}{1} = 1$ . The centre is  $(1, -2)$ ,  $a = 2$ ,  $b = 1$ .

25. Centre is  $(\frac{-D}{2A}, \frac{-E}{2B})$ , semi-axes are  $\sqrt{\frac{K}{A}}$ ,  $\sqrt{\frac{K}{B}}$ , in which  $K = -F + \frac{D^2}{4A} + \frac{E^2}{4B}$ .

26. The locus is the curve  $2xy - 7x + 4y = 0$ . If we change the origin to the point  $(h, k)$ , we can so choose the values of  $h$  and  $k$  as to get rid of the terms containing  $x$  and  $y$ . Making the change, we obtain

$$2xy + (2k - 7)x + (2h + 4)y - 7h + 4k + 2hk = 0.$$

If we choose  $h$  and  $k$  so that  $2h + 4 = 0$ , and  $2k - 7 = 0$ , that is, if we take  $h = -2$ ,  $k = \frac{7}{2}$ , the terms containing  $x$  and  $y$  vanish, and the equation becomes  $xy = -7$ . Hence we see (§ 182, Cor.) that the locus is an equilateral hyperbola, whose branches lie in the second and fourth quadrants, and that the new axes of co-ordinates are the asymptotes.

27. The equilateral hyperbola  $2xy = a^2$ .

28. Taking the base as axis of  $x$ , and the vertex of the smaller angle as origin, the locus consists of the axis of  $x$  and the hyperbola  $3x^2 - y^2 - 2ax = 0$ .

#### Ex. 40. Page 206.

1. The ellipse  $72x^2 + 48y^2 = 35$ .
2. The ellipse  $4x^2 + 2y^2 = 1$ .
3. The hyperbola  $32x^2 - 48y^2 = 9$ .
4. The ellipse  $9x^2 + 3y^2 = 32$ .
5. The hyperbola  $4x^2 - 4y^2 + 1 = 0$ .
6. The parabola  $y^2 = -\frac{4}{3}x$ .
7. The parabola  $y^2 = 2x\sqrt{2}$ .
8. The parabola  $y^2 = 3x\sqrt{2}$ .
9. The parabola  $y^2 = 2x$ .
10. The ellipse  $4x^2 + 9y^2 = 36$ .
11. The point  $(0, 0)$ .
12. The hyperbola  $4x^2 - 9y^2 = 36$ .
13. The straight lines  $y = x, y = -5$ .



**Ex. 42. Page 229.**

2. 2d; 5th; 6th; 7th; 3d; 4th; 8th.
5.  $5\sqrt{2}$ ;  $\sqrt{14}$ ;  $\sqrt{83}$ ;  $3\sqrt{2}$ ;  $4\sqrt{2}$ ;  $5\sqrt{2}$ ;  $\frac{1}{4}\sqrt{14}$ ;  $-\frac{3}{4}\sqrt{14}$ ;  $-\frac{1}{4}\sqrt{14}$ ;  $\frac{7}{3}\sqrt{83}$ ;  $-\frac{1}{3}\sqrt{83}$ ;  $-\frac{2}{3}\sqrt{83}$ .
6.  $\frac{1}{14}\sqrt{14}$ ;  $\frac{1}{7}\sqrt{14}$ ;  $\frac{3}{14}\sqrt{14}$ . The line is parallel to the radius vector of the point (1, 2, 3).
7. Parallel to the radius vector of the point (A, B, C).
8.  $60^\circ$  or  $120^\circ$ ;  $90^\circ$ ;  $60^\circ$  or  $120^\circ$ .

**Ex. 43. Page 235.**

1.  $\sqrt{3}$ ;  $\sqrt{3}$ ;  $2\sqrt{3}$ .
3.  $5\sqrt{2}$ .
5. Lines parallel to the radius vector of  $(3, -2, -5)$ .
6.  $\cos^{-1} \frac{1}{3}\sqrt{14}$ .
9.  $(1, \sqrt{3}, 2\sqrt{3})$ .
7.  $90^\circ$ .
12.  $(\frac{5}{2}, -\frac{3}{2}, \frac{5}{2})$ .
8.  $(4, \frac{1}{2}\pi, \frac{1}{2}\pi)$ .
13.  $(-2^{\frac{2}{3}}, -\frac{1}{2}, \frac{1}{2})$ .

**Ex. 44. Page 240.**

4.  $2x - 3y + \sqrt{3}z = 28$ .  
5.  $\frac{x}{4} - \frac{y}{3} - \frac{z}{7} = 1$ .  
6.  $6x + y - z = 5$ .

The general equation of a plane may be divided by any one of its coefficients; hence, except when  $D = 0$ , it may be put under the form  $A_1x + B_1y + C_1z = 1$ , which contains three arbitrary constants, and therefore may be made to satisfy three simple conditions.

7.  $\cos^{-1} \frac{1}{4}$
8.  $79^\circ 52'$
9.  $\cos^{-1} \frac{C}{\sqrt{A^2 + B^2 + C^2}}$
10. 2.25.
11.  $\cos^{-1} \frac{A}{\sqrt{A^2 + B^2 + C^2}}$
12.  $\frac{x}{a} + \frac{y}{b} - \frac{z}{c} = 1$ ;  
 $\frac{-2abc}{\sqrt{b^2c^2 + a^2c^2 + a^2b^2}}$
13.  $\cos^{-1} \frac{B}{\sqrt{A^2 + B^2 + C^2}}$
14.  $2x + 4y - z = 23$ .
15.  $\cos^{-1} \frac{C}{\sqrt{A^2 + B^2 + C^2}}$
16.  $3x - y + 2z = 4$ .
17.  $\cos^{-1} \frac{A}{\sqrt{A^2 + B^2 + C^2}}$
18.  $x + y + z = 6$ ;  $2\sqrt{3}$ .
19.  $3x - 4y + 7z + 13 = 0$ .
20.  $2x + 5y - z = 9$ .

## Ex. 45. Page 247.

3. It passes through  $(\frac{1}{3}, 0, -\frac{2}{3})$ , and is parallel to the radius vector of  $(4, 5, 3)$ .

$$5. \frac{x-1}{2} = \frac{y-2}{2} = \frac{z-3}{-2}$$

$$6. (4, 5, 0), (0, 1, 4), (-1, 0, 5).$$

$$7. 2y + 5z = 10.$$

$$8. 2x - 5y + 1 = 0, 2x + 5z + 1 = 0, y + z = 0.$$

$$11. \cos^{-1} .1.$$

$$16. \frac{x-3}{-3} = \frac{y+7}{5} = \frac{z+5}{-6}.$$

$$12. 14^{\circ} 57' 46''.$$

$$17. \frac{x-2}{3} = \frac{y+4}{-6} = \frac{z+6}{2}.$$

$$13. \cos^{-1} \frac{1}{3}\sqrt{2}.$$

$$15. y = -2x - 1, z = 3x + 5.$$

$$18. \sin^{-1} \frac{2}{3}.$$

$$19. \frac{\frac{x-p}{m}}{\sqrt{1+m^2+n^2}} = \frac{\frac{y-q}{n}}{\sqrt{1+m^2+n^2}} = \frac{\frac{z}{1}}{\sqrt{1+m^2+n^2}},$$

in which the denominators equal  $\cos \alpha, \cos \beta, \cos \gamma$  respectively.

## Ex. 46. Page 263.

1. The planes  $x = -4, x = -1, x = 2$ ; the planes  $y = -2, y = 1, y = 3$ ; the planes  $z = 0, z = -m$ .

2. Answer to the first, the parabolic cylinder whose elements are parallel to the axis of  $z$ , and whose trace on the plane  $xy$  is the parabola  $y^2 = 8x$ .

$$3. 5x^2 + 13y^2 = 56, 5z^2 - y^2 = 8, x^2 + 13z^2 = 32.$$

4.  $x^2 + y^2 = \frac{4}{3}, z = \pm \frac{2}{3}\sqrt{42}$ ; hence the curves are two circles in the planes  $z = \pm \frac{2}{3}\sqrt{42}$ , whose centres are in the axis of  $z$  and whose radii are  $\sqrt{\frac{4}{3}}$  each.

$$5. a = 3, b = 2, c = \frac{1}{3}\sqrt{5}.$$

$$6. \text{Two circles whose radii are } \sqrt{13} \text{ each.}$$

$$8. 9(x^2 + y^2) = (z - 5)^2; x^2 + y^2 = \frac{2}{9}z^2.$$

$$9. 25x^2 + 25y^2 - 9z^2 + 90z = 225.$$

$$10. x^2 + y^2 - \frac{3}{2}z - \frac{5}{2} = 0.$$

$$11. y^2 + z^2 - 8x = 0.$$

$$12. x^2 + y^2 - \frac{1}{4}z^2 - 3z = 9; (0, 0, -6).$$

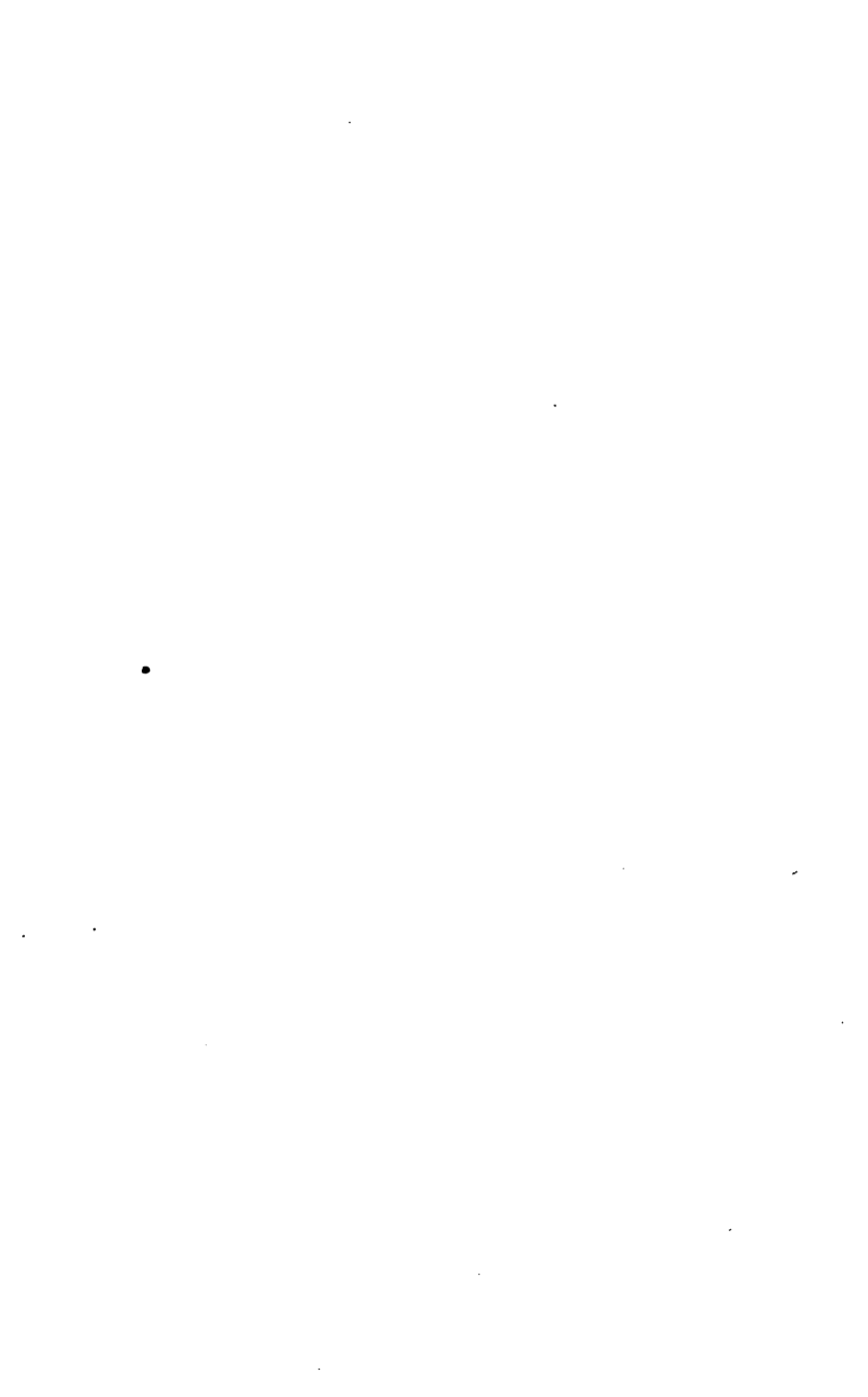
$$13. 9(x^2 + y^2) + 4z^2 = 36.$$

$$14. 16(x^2 + y^2) + 9z^2 = 144.$$

$$15. 4(x^2 + y^2) - 9z^2 = 36.$$

$$16. x^2z^4 + y^2z^4 = 1, x^2 + y^2 - \frac{1}{2}z^2 = 0.$$

$$17. a = 5\sqrt{3}, b = \frac{2}{3}\sqrt{6}.$$



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